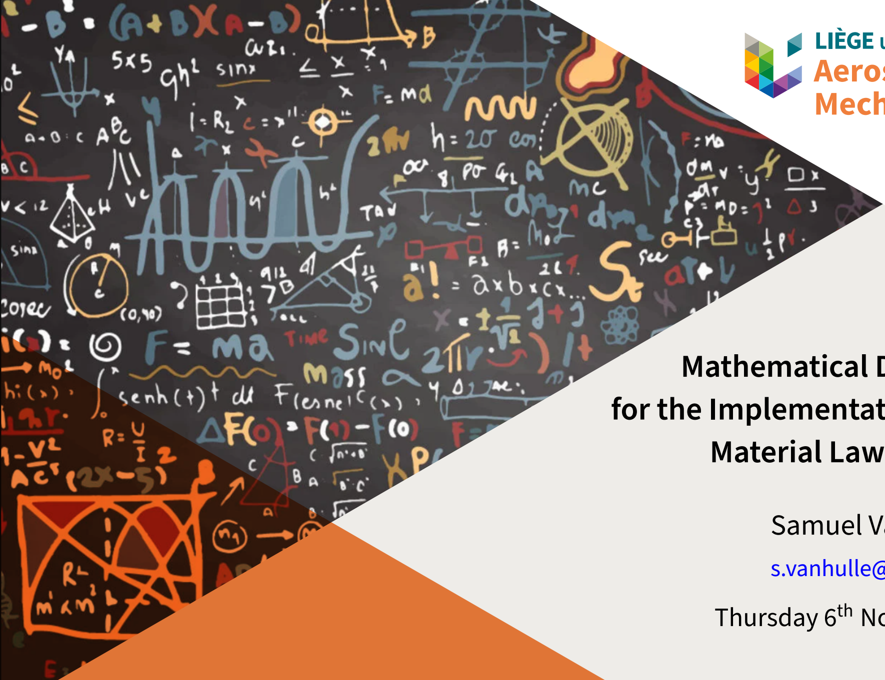

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Mathematical Developments for the Implementation of Hyperelastic Material Laws in Metafor

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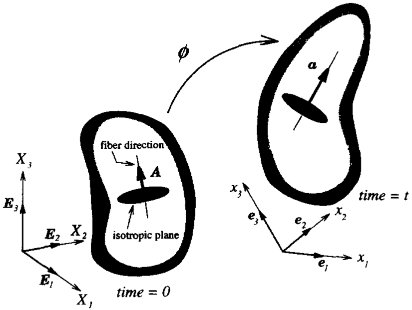
General Expressions

Considering a transversely isotropic solid under large displacements and strains, let $\mathbf{a}_0$ denote the principal fiber direction of orthotropy in the undeformed configuration.

Let's also introduce the second order structural tensor $\mathbf{M}=\mathbf{a}_0 \otimes \mathbf{a}_0$.



[Bonet and Burton, 1998]

General Expressions

The total Helmholtz free energy writes

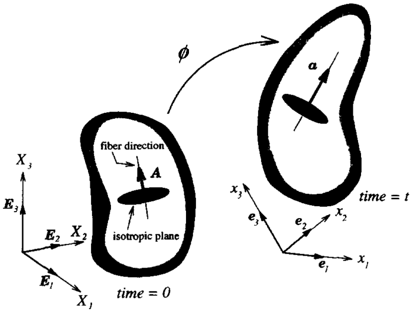
$$\psi = \psi(I_1, I_2, J, I_4, I_5), \tag{1}$$

with the introduction of two additional invariants of the right Cauchy-Green tensor $\mathbf{C}$ defined by

$$I_4 = \mathrm{tr}\,(\mathbf{CM}) = \mathbf{C} : \mathbf{M} = \mathbf{a}_0 \cdot \mathbf{C}\mathbf{a}_0 \tag{2}$$

$$I_5 = \mathrm{tr}\,\left(\mathbf{C}^2\mathbf{M}\right) = \mathbf{C}^2 : \mathbf{M} = \mathbf{a}_0 \cdot \mathbf{C}^2\mathbf{a}_0 \tag{3}$$

I_4 corresponds to the stretch in the fiber direction (λ_F) whereas I_5 is related to the cross-section reduction in the direction orthogonal to $\mathbf{a}_0$.



[Bonet and Burton, 1998]

PK2 Stress Tensor

The general expression of the PK2 stress tensor $\mathbf{S}$ writes

$$\mathbf{S} = 2\frac{\partial \psi}{\partial \mathbf{C}} = 2\left[\frac{\partial \psi}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \psi}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \psi}{\partial J}\frac{\partial J}{\partial \mathbf{C}} + \frac{\partial \psi}{\partial I_4}\frac{\partial I_4}{\partial \mathbf{C}} + \frac{\partial \psi}{\partial I_5}\frac{\partial I_5}{\partial \mathbf{C}}\right] \tag{4}$$

Using the following expressions for the derivatives of the invariants

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{I} \;\Big|\; \frac{\partial I_2}{\partial \mathbf{C}} = I_1\mathbf{I} - \mathbf{C} \;\Big|\; \frac{\partial J}{\partial \mathbf{C}} = \frac{J}{2}\mathbf{C}^{-1} \;\Big|\; \frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{M} \;\Big|\; \frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{MC} + \mathbf{CM} \tag{5}$$

the PK2 stress tensor can be written as

$$\mathbf{S} = 2\left[\left(\frac{\partial \psi}{\partial I_1} + \frac{\partial \psi}{\partial I_2}I_1\right)\mathbf{I} - \frac{\partial \psi}{\partial I_2}\mathbf{C} + \frac{\partial \psi}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial \psi}{\partial I_4}\mathbf{M} + \frac{\partial \psi}{\partial I_5}(\mathbf{MC} + \mathbf{CM})\right] \tag{6}$$

Isotropic/Anisotropic Split of Potential

Without loss of generality, the potential ψ can be decomposed into an isotropic contribution ψ_i and an orthotropic contribution ψ_a as

$$\psi(I_1, I_2, J, I_4, I_5) = \psi_i(I_1, I_2, J) + \psi_a(I_1, I_2, J, I_4, I_5), \tag{7}$$

and consequently, the PK2 stress tensor can also be decomposed in an isotropic contribution $\mathbf{S}_i$ and an orthotropic contribution $\mathbf{S}_a$ as

$$\mathbf{S} = \mathbf{S}_i + \mathbf{S}_a = 2\left[\frac{\partial \psi_i}{\partial \mathbf{C}} + \frac{\partial \psi_a}{\partial \mathbf{C}}\right] \tag{8}$$

Deviatoric/Volumetric Split of Deformation Gradient

Using the deviatoric/volumetric split of the total deformation gradient **F** from [Flory, 1961]

$$\mathbf{F} = \hat{\mathbf{F}}\bar{\mathbf{F}}, \quad \text{with} \quad \begin{cases} \hat{\mathbf{F}} = (\det\mathbf{F})^{\frac{1}{3}}\,\mathbf{I} = J^{\frac{1}{3}}\mathbf{I} \\ \bar{\mathbf{F}} = (\det\mathbf{F})^{-\frac{1}{3}}\,\mathbf{F} = J^{-\frac{1}{3}}\mathbf{F} \end{cases} \tag{9}$$

the strain-energy density function ψ can be written in terms of the reduced invariants

$$\psi(I_1, I_2, J, I_4, I_5) = \bar{\psi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5) = \bar{\psi}_i(\bar{I}_1, \bar{I}_2, \bar{I}_3) + \bar{\psi}_a(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5), \tag{10}$$

Starting from Eqn. 6, the PK2 stress tensor **S** now writes

$$\mathbf{S} = 2J^{-\frac{2}{3}}\left[\left(\frac{\partial\bar{\psi}}{\partial\bar{I}_1} + \frac{\partial\bar{\psi}}{\partial\bar{I}_2}\bar{I}_1\right)\mathbf{I} - \frac{\partial\bar{\psi}}{\partial\bar{I}_2}\bar{\mathbf{C}} + \frac{\partial\bar{\psi}}{\partial J}\frac{J}{2}\bar{\mathbf{C}}^{-1} + \frac{\partial\bar{\psi}}{\partial\bar{I}_4}\mathbf{M} + \frac{\partial\bar{\psi}}{\partial\bar{I}_5}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right)\right] \tag{11}$$

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Transversely Isotropic St. Venant Model

For the isotropic contribution ψ_i, the standard St. Venant potential writes

$$\psi_i = \frac{1}{2}\lambda \left(\mathrm{tr}\mathbf{E}^{GL}\right)^2 + \mu\mathbf{E}^{GL} : \mathbf{E}^{GL}, \tag{12}$$

where $\mathbf{E}^{GL}$ is the Green-Lagrange strain tensor, λ and μ (or G) are related to the engineering material constants as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \Big| \quad \mu = G = \frac{E}{2(1+\nu)}, \tag{13}$$

with Young modulus E and Poisson ratio ν.

Transversely Isotropic St. Venant Model

The simplest transversely isotropic potential ψ_a that satisfies the requirements of resulting in a constant elasticity tensor while providing enough material constants to reproduce the small strain model writes [Bonet and Burton, 1998]

$$\psi_a = \left[\alpha + \beta\left(I_1 - 3\right) + \gamma\left(I_4 - 1\right)\right]\left(I_4 - 1\right) - \frac{1}{2}\alpha\left(I_5 - 1\right), \tag{14}$$

For the transversely isotropic St. Venant model, the PK2 stress tensor $\mathbf{S}$ finally writes

$$\begin{aligned} \mathbf{S} = \mathbf{S}_i + \mathbf{S}_a &= 2\frac{\partial\psi_i}{\partial\mathbf{C}} + 2\frac{\partial\psi_a}{\partial\mathbf{C}} \\ &= \lambda \mathrm{tr}\left(\mathbf{E}^{GL}\right)\mathbf{I} + 2\mu\mathbf{E}^{GL} + 2\beta(I_4 - 1)\mathbf{I} + 2\left[\alpha + \beta\left(I_1 - 3\right) + 2\gamma\left(I_4 - 1\right)\right]\mathbf{M} - \alpha\left(\mathbf{CM} + \mathbf{MC}\right) \end{aligned} \tag{15}$$

Identification of Material Parameters

The different constants from the transversely isotropic St.Venant material are related to the engineering material constants in the principal (fiber) direction (1) and transverse directions (2, 3) [Abd Latif *et al*, 2007]

$$\mu = \frac{E_2}{2\left(1+\nu_{23}\right)} \quad \Big| \quad \lambda = -\frac{E_2\left(E_2\nu_{12}^2 + E_1\nu_{23}\right)}{\left(2E_2\nu_{12}^2 + E_1\left[\nu_{23}-1\right]\right)\left(1+\nu_{23}\right)} \tag{16}$$

$$\alpha = \frac{1}{4}\left(\frac{E_2}{1+\nu_{23}} - 2G_{12}\right) \quad \Big| \quad \beta = \frac{E_2}{4}\left(\frac{E_1 - 2E_1\nu_{12}}{2E_2\nu_{12}^2 + E_1\left(\nu_{23}-1\right)} + \frac{1}{1+\nu_{23}}\right) \tag{17}$$

$$\gamma = \frac{1}{16}\left(2E_1 - 4G_{12} - \frac{E_1E_2\left(1-2\nu_{12}\right)^2}{2E_2\nu_{12}^2 + E_1\left(\nu_{23}-1\right)} - \frac{E_2}{1+\nu_{23}}\right) \tag{18}$$

Identification of Material Parameters

Note that when the material is purely isotropic, we have

$$E_1 = E_2 = E \quad \Big| \quad \nu_{12} = \nu_{23} = \nu \quad \Big| \quad G_{12} = G = \mu \tag{19}$$

leading to the following constant values

$$\mu = \frac{E}{2(1+\nu)} \quad \Big| \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \Big| \quad \alpha = 0 \quad \Big| \quad \beta = 0 \quad \Big| \quad \gamma = 0 \tag{20}$$

which is equivalent to the isotropic St. Venant model.

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Isotropic Strain-Energy Density Function

The (general) strain-energy density for an isotropic Neo-Hookean material writes

$$\psi_i(\bar{I}_1, J) = \frac{\mu}{2}(\bar{I}_1 - 3) + \frac{\lambda}{2} f(J) = C_1(\bar{I}_1 - 3) + K f(J) \tag{21}$$

Note that compared to [Bonet and Burton, 1998], the gauge term "$-\mu \ln J$" is not required due to the use of reduced invariants.

For [Bonet and Burton, 1998], the volumetric part of the strain-energy density function writes

$$K f(J) = \frac{\lambda}{2}(J - 1)^2 \tag{22}$$

Isotropic PK2 and Cauchy Stress Tensors

The isotropic PK2 stress tensor $\mathbf{S}_{iso}$ is obtained from the differentiation of ψ_i with respect to $\bar{\mathbf{C}}$ as in Eqn. 11

$$\begin{aligned}\mathbf{S}_{iso} &= 2J^{-\frac{2}{3}}\left[\frac{\partial\bar{\psi}_i}{\partial\bar{I}_1}\mathbf{I} + \frac{\partial\bar{\psi}_i}{\partial J}\frac{J}{2}\bar{\mathbf{C}}^{-1}\right] = 2J^{-\frac{2}{3}}\left[\frac{\mu}{2}\mathbf{I} + \left(-\frac{\mu}{3}J^{-\frac{5}{3}}I_1 + \lambda[J-1]\right)\frac{J}{2}\bar{\mathbf{C}}^{-1}\right] \\ &= J^{-\frac{2}{3}}\left[\mu\mathbf{I} + \left(-\frac{\mu}{3}\bar{I}_1 + \lambda J[J-1]\right)\bar{\mathbf{C}}^{-1}\right]\end{aligned} \tag{23}$$

The isotropic Cauchy stress tensor $\boldsymbol{\sigma}_{iso}$ is obtained from the push-forward of Eqn. 23

$$\begin{aligned}\boldsymbol{\sigma}_{iso} = \frac{1}{J}\mathbf{F}\mathbf{S}_{iso}\mathbf{F}^T &= \frac{J^{-\frac{2}{3}}}{J}\left[\mu\mathbf{F}\mathbf{I}\mathbf{F}^T + \left(-\frac{\mu}{3}\bar{I}_1 + \lambda J[J-1]\right)\mathbf{F}\bar{\mathbf{C}}^{-1}\mathbf{F}^T\right] \\ &= \frac{\mu}{J}J^{-\frac{2}{3}}\mathbf{B} + \frac{1}{J}\left(-\frac{\mu}{3}\bar{I}_1 + \lambda J[J-1]\right)\frac{J^{-\frac{2}{3}}}{J^{-\frac{2}{3}}}\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T \\ &= \frac{\mu}{J}\bar{\mathbf{B}} - \frac{1}{3}\frac{\mu}{J}\bar{I}_1\mathbf{I} + \lambda(J-1)\mathbf{I} = \frac{\mu}{J}\mathrm{dev}\left(\bar{\mathbf{B}}\right) + \lambda(J-1)\mathbf{I}\end{aligned} \tag{24}$$

Isotropic Deviatoric Stress and Pressure

From Eqn. 24, the isotropic deviatoric stress contribution $\mathbf{s}_{iso}$ writes

$$\mathbf{s}_{iso} = \mathrm{dev}\left(\boldsymbol{\sigma}_{iso}\right) = \frac{\mu}{J}\mathrm{dev}\left(\bar{\mathbf{B}}\right), \tag{25}$$

and the isotropic pressure contribution p_{iso} writes

$$p_{iso} = \frac{\mathrm{tr}\left(\boldsymbol{\sigma}_{iso}\right)}{3} = \lambda(J-1). \tag{26}$$

Isotropic Material Elasticity Tensor (PK2)

The isotropic material elasticity tensor $\mathbb{H}^{\mathbf{S}_{iso}}$ is obtained from the differentiation of Eqn. 23

$$\mathbb{H}^{\mathbf{S}_{iso}} = 2\frac{\partial \mathbf{S}_{iso}}{\partial \mathbf{C}} = 2\left(\frac{\partial \mathbf{S}^{vol}_{iso}}{\partial \mathbf{C}} + \frac{\partial \mathbf{S}^{dev}_{iso}}{\partial \mathbf{C}}\right) = \mathbb{H}^{\mathbf{S}_{iso}}_{vol} + \mathbb{H}^{\mathbf{S}_{iso}}_{dev} \tag{27}$$

using the split in the deviatoric part of the material elasticity tensor $\mathbb{H}^{\mathbf{S}_{iso}}_{dev}$ and the volumetric part of the material elasticity tensor $\mathbb{H}^{\mathbf{S}_{iso}}_{vol}$. The volumetric part will not be computed since it is not relevant for Metafor.

$$\begin{aligned}
\mathbb{H}^{\mathbf{S}_{iso}}_{dev} =& 2\frac{\partial}{\partial \mathbf{C}}\left(\mu J^{-\frac{2}{3}}\left[\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]\right) \\
=& 2\mu\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \frac{\partial}{\partial \mathbf{C}}\left(J^{-\frac{2}{3}}\right) + 2\mu J^{-\frac{2}{3}}\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \\
=& 2\mu\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \frac{\partial}{\partial J}\left(J^{-\frac{2}{3}}\right)\frac{\partial J}{\partial \mathbf{C}} + 2\mu J^{-\frac{2}{3}}\left[\frac{\partial \mathbf{I}}{\partial \mathbf{C}} - \frac{1}{3}\frac{\partial}{\partial C}\left(\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\right]
\end{aligned}$$

Isotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
&=2\mu\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(-\frac{2}{3}J^{-\frac{5}{3}}\frac{J}{2}\mathbf{C}^{-1}\right)+2\mu J^{-\frac{2}{3}}\left[0-\frac{1}{3}\left(\bar{\mathbf{C}}^{-1}\otimes\frac{\partial\bar{I}_1}{\partial I_1}\mathbf{I}+\bar{\mathbf{C}}^{-1}\otimes\frac{\partial\bar{I}_1}{\partial J}\frac{J}{2}\mathbf{C}^{-1}+\bar{I}_1\frac{\partial\bar{\mathbf{C}}^{-1}}{\partial\mathbf{C}}\right)\right]\\
&=-\frac{2}{3}\mu J^{-\frac{4}{3}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}-\frac{2}{3}\mu J^{-\frac{2}{3}}\left(\bar{\mathbf{C}}^{-1}\otimes J^{-\frac{2}{3}}\mathbf{I}+\bar{\mathbf{C}}^{-1}\otimes\left[-\frac{1}{3}\bar{I}_1\right]\mathbf{C}^{-1}+\bar{I}_1\frac{\partial\bar{\mathbf{C}}^{-1}}{\partial\bar{\mathbf{C}}}:\frac{\partial\bar{\mathbf{C}}}{\partial\mathbf{C}}\right)\\
&=-\frac{2}{3}\mu J^{-\frac{4}{3}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}-\frac{2}{3}\mu J^{-\frac{4}{3}}\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]+\bar{I}_1\frac{\partial\bar{\mathbf{C}}^{-1}}{\partial\bar{\mathbf{C}}}:\left[\mathbb{I}^{sym}-\frac{1}{3}\bar{\mathbf{C}}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
&=-\frac{2}{3}\mu J^{-\frac{4}{3}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}-\frac{2}{3}\mu J^{-\frac{4}{3}}\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]-\bar{I}_1\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
&=-\frac{2}{3}\mu J^{-\frac{4}{3}}\left(\left[\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]-\bar{I}_1\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
\tag{28}
$$

Isotropic Material Elasticity Tensor (Cauchy)

The deviatoric part of the isotropic material elasticity tensor with respect to Cauchy stresses $\mathbb{H}^{\boldsymbol{\sigma}_{iso}}_{dev}$ is obtained from the push-forward operation of Eqn. 28 to the current configuration

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}_{iso}}_{dev} &= \frac{1}{J}\mathbf{F}\left(\mathbf{F}\mathbb{H}^{\mathbf{S}_{iso}}_{dev}\mathbf{F}^T\right)\mathbf{F}^T \\
&= -\frac{2}{3}\frac{\mu}{J}\left(\left[\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right]\otimes\mathbf{I} + \mathbf{I}\otimes\left[\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right] - \bar{I}_1\left[\mathbf{I}\odot\mathbf{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right]\right) \\
&= \frac{2}{3}\frac{\mu}{J}\bar{I}_1\left(\mathbb{I}^{sym} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right) - \frac{2}{3}\frac{\mu}{J}\left(\operatorname{dev}\left[\bar{\mathbf{B}}\right]\otimes\mathbf{I} + \mathbf{I}\otimes\operatorname{dev}\left[\bar{\mathbf{B}}\right]\right) && (29) \\
&= \frac{2}{3}\frac{\mu}{J}\bar{I}_1\left(\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right) - \frac{2}{3}\frac{\mu}{J}\left(\bar{\mathbf{B}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{B}}\right) && (30)
\end{aligned}
$$

This expression presents a major and a minor symmetry as expected.

Isotropic Part of the Material Elasticity Tensor (Cauchy)

Using index notation, Eqn. 30 finally writes

$$\mathbb{H}^{\boldsymbol{\sigma}_{iso}}_{ijkl} = \frac{2}{3}\frac{\mu}{J}\bar{B}_{mm}\left(\frac{1}{2}\left[\delta_{il}\delta_{jk} + \delta_{ik}\delta_{il} + \frac{1}{3}\delta_{ij}\delta_{kl}\right]\right) - \frac{2}{3}\frac{\mu}{J}\left(\bar{B}_{ij}\delta_{kl} + \delta ij\bar{B}_{kl}\right) \tag{31}$$

Anisotropic Strain-Energy Density Function

For the anisotropic contribution to the strain-energy density function, Eqn. 14 from [Bonet and Burton, 1998] is re-written in terms of the reduced invariants, hence

$$\psi_a = \left[\alpha + \beta\left(\bar{I}_1 - 3\right) + \gamma\left(\bar{I}_4 - 1\right)\right]\left(\bar{I}_4 - 1\right) - \frac{1}{2}\alpha\left(\bar{I}_5 - 1\right), \tag{32}$$

The anisotropic PK2 stress tensor $\mathbf{S}_{ani}$ is obtained from the differentiation of ψ_a with respect to $\bar{\mathbf{C}}$ as in Eqn. 11

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}}\left[\frac{\partial\bar{\psi}}{\partial\bar{I}_1}\mathbf{I} + \frac{\partial\bar{\psi}}{\partial J}\frac{J}{2}\bar{\mathbf{C}}^{-1} + \frac{\partial\bar{\psi}}{\partial\bar{I}_4}\mathbf{M} + \frac{\partial\bar{\psi}}{\partial\bar{I}_5}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right)\right] \tag{33}$$

Anisotropic PK2 Stress Tensor

Managing each differential term separately

$$\frac{\partial \psi_a}{\partial \bar{I}_1} = \beta(\bar{I}_4 - 1) \tag{34}$$

$$\frac{\partial \psi_a}{\partial \bar{I}_4} = \alpha + \beta(\bar{I}_1 - 3) + 2\gamma(\bar{I}_4 - 1) \tag{35}$$

$$\frac{\partial \psi_a}{\partial \bar{I}_5} = -\frac{\alpha}{2} \tag{36}$$

$$\begin{aligned}
\frac{\partial \psi_a}{\partial J} &= -\frac{2}{3}J^{-\frac{5}{3}}I_4\left[\alpha + \beta\left(\bar{I}_1 - 3\right) + \gamma\left(\bar{I}_4 - 1\right)\right] + \left[-\frac{2}{3}\beta J^{-\frac{5}{3}}I_1 - \frac{2}{3}\gamma J^{-\frac{5}{3}}I_4\right]\left(\bar{I}_4 - 1\right) + \frac{4}{3}\frac{\alpha}{2}J^{-\frac{7}{3}}I_5 \\
&= -\frac{2}{3}\frac{1}{J}\bar{I}_4\left[\alpha + \beta\left(\bar{I}_1 - 3\right) + \gamma\left(\bar{I}_4 - 1\right)\right] - \frac{2}{3}\frac{1}{J}\left[\beta\bar{I}_1 + \gamma\bar{I}_4\right]\left(\bar{I}_4 - 1\right) + \frac{2}{3}\frac{\alpha}{J}\bar{I}_5
\end{aligned} \tag{37}$$

Anisotropic PK2 Stress Tensor

and injecting those terms in Eqn. 33

$$
\begin{aligned}
\mathbf{S}_{ani} =& 2J^{-\frac{2}{3}} \Bigg[\beta(\bar{I}_4 - 1)\mathbf{I} + \Bigg(-\frac{2}{3}\frac{1}{J}\bar{I}_4 \left[\alpha + \beta \left(\bar{I}_1 - 3 \right) + \gamma \left(\bar{I}_4 - 1 \right) \right] - \frac{2}{3}\frac{1}{J} \left[\beta \bar{I}_1 + \gamma \bar{I}_4 \right] \left(\bar{I}_4 - 1 \right) \\
& + \frac{2}{3}\frac{\alpha}{J}\bar{I}_5 \Bigg) \frac{J}{2}\bar{\mathbf{C}}^{-1} + \left(\alpha + \beta \left[\bar{I}_1 - 3 \right] + 2\gamma \left[\bar{I}_4 - 1 \right] \right) \mathbf{M} - \frac{\alpha}{2} \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} \right) \Bigg] \\
=& 2J^{-\frac{2}{3}} \Bigg[\beta(\bar{I}_4 - 1)\mathbf{I} + \Bigg(-\frac{1}{3}\bar{I}_4 \left[\alpha + \beta \left(\bar{I}_1 - 3 \right) + \gamma \left(\bar{I}_4 - 1 \right) \right] - \frac{1}{3} \left[\beta \bar{I}_1 + \gamma \bar{I}_4 \right] \left(\bar{I}_4 - 1 \right) \\
& + \frac{1}{3}\alpha\bar{I}_5 \Bigg) \bar{\mathbf{C}}^{-1} + \left(\alpha + \beta \left[\bar{I}_1 - 3 \right] + 2\gamma \left[\bar{I}_4 - 1 \right] \right) \mathbf{M} - \frac{\alpha}{2} \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} \right) \Bigg] \\
=& 2J^{-\frac{2}{3}} \Bigg[\beta(\bar{I}_4 - 1) \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1} \right) + \left(\alpha + \beta \left[\bar{I}_1 - 3 \right] + 2\gamma \left[\bar{I}_4 - 1 \right] \right) \left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1} \right) \\
& - \frac{\alpha}{2} \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1} \right) \Bigg]
\end{aligned}
\tag{38}
$$

Anisotropic Cauchy Stress Tensor

The anisotropic Cauchy stress tensor $\boldsymbol{\sigma}_{ani}$ is obtained from the push-forward of Eqn. 38

$$
\begin{aligned}
\boldsymbol{\sigma}_{ani} = \frac{1}{J}\mathbf{F}\mathbf{S}_{ani}\mathbf{F}^T =& 2J^{-\frac{5}{3}}\left[\beta(\bar{I}_4-1)\left(\mathbf{F}\mathbf{I}\mathbf{F}^T-\frac{1}{3}\bar{I}_1\mathbf{F}\bar{\mathbf{C}}^{-1}\mathbf{F}^T\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\mathbf{F}\mathbf{M}\mathbf{F}^T-\frac{1}{3}\bar{I}_4\mathbf{F}\bar{\mathbf{C}}^{-1}\mathbf{F}^T\right)\right.\\
&\left.-\frac{\alpha}{2}\left(\mathbf{F}\mathbf{M}\bar{\mathbf{C}}\mathbf{F}^T+\mathbf{F}\bar{\mathbf{C}}\mathbf{M}\mathbf{F}^T-\frac{2}{3}\bar{I}_5\mathbf{F}\bar{\mathbf{C}}^{-1}\mathbf{F}^T\right)\right]\\
=& 2J^{-\frac{5}{3}}\left[\beta(\bar{I}_4-1)\left(\mathbf{B}-\frac{1}{3}\bar{I}_1J^{\frac{2}{3}}\mathbf{I}\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\mathbf{N}-\frac{1}{3}\bar{I}_4J^{\frac{2}{3}}\mathbf{I}\right)\right.\\
&\left.-\frac{\alpha}{2}\left(\mathbf{F}\mathbf{M}\bar{\mathbf{F}}^T\bar{\mathbf{F}}\mathbf{F}^T+\mathbf{F}\bar{\mathbf{F}}^T\bar{\mathbf{F}}\mathbf{M}\mathbf{F}^T-\frac{2}{3}\bar{I}_5J^{\frac{2}{3}}\mathbf{I}\right)\right]\\
=& \frac{2}{J}\left[\beta(\bar{I}_4-1)\left(\bar{\mathbf{B}}-\frac{1}{3}\bar{I}_1\mathbf{I}\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{N}}-\frac{1}{3}\bar{I}_4\mathbf{I}\right)\right.\\
&\left.-\frac{\alpha}{2}\left(\bar{\mathbf{F}}\mathbf{M}\bar{\mathbf{F}}^T\bar{\mathbf{F}}\bar{\mathbf{F}}^T+\bar{\mathbf{F}}\bar{\mathbf{F}}^T\bar{\mathbf{F}}\mathbf{M}\bar{\mathbf{F}}^T-\frac{2}{3}\bar{I}_5\mathbf{I}\right)\right]
\end{aligned}
$$

Anisotropic Cauchy Stress Tensor

$$
\begin{aligned}
=&\frac{2}{J}\Bigg[\beta\left(\bar{I}_4-1\right)\left(\bar{\mathbf{B}}-\frac{1}{3}\bar{I}_1\mathbf{I}\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{N}}-\frac{1}{3}\bar{I}_4\mathbf{I}\right)\\
&-\frac{\alpha}{2}\left(\bar{\mathbf{N}}\bar{\mathbf{B}}+\bar{\mathbf{B}}\bar{\mathbf{N}}-\frac{2}{3}\bar{I}_5\mathbf{I}\right)\Bigg]\\
=&\frac{2}{J}\Bigg[\beta\left(\bar{I}_4-1\right)\operatorname{dev}\left(\bar{\mathbf{B}}\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\operatorname{dev}\left(\bar{\mathbf{N}}\right)\\
&-\frac{\alpha}{2}\left(\operatorname{dev}\left(\bar{\mathbf{N}}\bar{\mathbf{B}}\right)+\operatorname{dev}\left(\bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right)\Bigg]
\end{aligned}
\tag{39}
$$

Eqn. 39 only contributes to the deviatoric stress $\mathbf{s}$.

Anisotropic Material Elasticity Tensor (PK2)

The anisotropic material elasticity tensor $\mathbb{H}^{\mathbf{S}_{ani}}$ is obtained from the differentiation of Eqn. 38

$$
\begin{aligned}
\mathbb{H}^{\mathbf{S}_{ani}} = 2\frac{\partial \mathbf{S}_{ani}}{\partial \mathbf{C}} =& 2\frac{\partial}{\partial \mathbf{C}}\left(2J^{-\frac{2}{3}}\left[\beta(\bar{I}_4 - 1)\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) + \left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right.\right.\\
&\left.\left. - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right]\right)\\
=& 4\left[\beta(\bar{I}_4 - 1)\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) + \left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right.\\
&\left. - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right] \otimes \frac{\partial}{\partial \mathbf{C}}\left(J^{-\frac{2}{3}}\right) + 4J^{-\frac{2}{3}}\frac{\partial}{\partial \mathbf{C}}\left(\beta[\bar{I}_4 - 1]\left[\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]\right)\\
& + 4J^{-\frac{2}{3}}\frac{\partial}{\partial \mathbf{C}}\left(\left[\alpha + \beta\left(\bar{I}_1 - 3\right) + 2\gamma\left(\bar{I}_4 - 1\right)\right]\left[\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right)\\
& - 2\alpha J^{-\frac{2}{3}}\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)
\end{aligned} \tag{40}
$$

For clarity, each terms from Eqn. 40 are derived separately hereafter.

Anisotropic Material Elasticity Tensor (PK2)

$$\boxed{\frac{\partial}{\partial \mathbf{C}}\left(J^{-\frac{2}{3}}\right)} = \frac{\partial}{\partial J}\left(J^{-\frac{2}{3}}\right)\frac{\partial J}{\partial \mathbf{C}} = -\frac{2}{3}J^{-\frac{5}{3}}\frac{J}{2}\mathbf{C}^{-1} = -\frac{1}{3}J^{-\frac{4}{3}}\bar{\mathbf{C}}^{-1} \tag{41}$$

$$\boxed{\frac{\partial}{\partial \mathbf{C}}\left(\beta[\bar{I}_4 - 1]\left[\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]\right)}$$

$$\begin{aligned}
=&\beta\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\frac{\partial}{\partial \mathbf{C}}\left(\bar{I}_4 - 1\right) + \beta\left(\bar{I}_4 - 1\right)\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\\
=&\beta\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(\frac{\partial(\bar{I}_4 - 1)}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial(\bar{I}_4 - 1)}{\partial I_4}\mathbf{M}\right) + \beta\left(\bar{I}_4 - 1\right)\left(\frac{\partial \mathbf{I}}{\partial \mathbf{C}} - \frac{1}{3}\left[\bar{\mathbf{C}}^{-1}\otimes\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \bar{I}_1\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \mathbf{C}}\right]\right)\\
=&\beta\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(-\frac{2}{3}J^{-\frac{5}{3}}I_4\frac{J}{2}\mathbf{C}^{-1} + J^{-\frac{2}{3}}\mathbf{M}\right) - \frac{1}{3}\beta\left(\bar{I}_4 - 1\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[\frac{\partial \bar{I}_1}{\partial I_1}\mathbf{I} + \frac{\partial \bar{I}_1}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right] + \bar{I}_1\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}} : \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}\right)
\end{aligned}$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
=&\beta\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(-\frac{1}{3}J^{-\frac{2}{3}}I_4\mathbf{C}^{-1}+J^{-\frac{2}{3}}\mathbf{M}\right)-\frac{1}{3}\beta\left(\bar{I}_4-1\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[J^{-\frac{2}{3}}\mathbf{I}-\frac{2}{3}J^{-\frac{5}{3}}\frac{J}{2}I_1\mathbf{C}^{-1}\right]\right.\\
&\left.+J^{-\frac{2}{3}}\bar{I}_1\frac{\partial\bar{\mathbf{C}}^{-1}}{\partial\bar{\mathbf{C}}}:\left[\mathbb{I}^{sym}-\frac{1}{3}\bar{\mathbf{C}}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
=&\beta J^{-\frac{2}{3}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)-\frac{1}{3}\beta\left(\bar{I}_4-1\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[J^{-\frac{2}{3}}\mathbf{I}-\frac{1}{3}J^{-\frac{2}{3}}I_1\mathbf{C}^{-1}\right]\right.\\
&\left.-J^{-\frac{2}{3}}\bar{I}_1\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
=&\beta J^{-\frac{2}{3}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)-\frac{1}{3}\beta J^{-\frac{2}{3}}\left(\bar{I}_4-1\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]\right.\\
&\left.-\bar{I}_1\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
\tag{42}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$\boxed{\frac{\partial}{\partial \mathbf{C}}\left(\left[\alpha+\beta\left(\bar{I}_1-3\right)+2\gamma\left(\bar{I}_4-1\right)\right]\left[\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right)}$$

$$= \left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\frac{\partial}{\partial \mathbf{C}}\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)$$

$$= \left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\beta\frac{\partial(\bar{I}_1-3)}{\partial \mathbf{C}}+2\gamma\frac{\partial(\bar{I}_4-1)}{\partial \mathbf{C}}\right)+\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\frac{\partial \mathbf{M}}{\partial \mathbf{C}}\right.$$

$$\left.-\frac{1}{3}\left[\bar{\mathbf{C}}^{-1}\otimes\frac{\partial \bar{I}_4}{\partial \mathbf{C}}+\bar{I}_4\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \mathbf{C}}\right]\right)$$

$$= \left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\beta\left[\frac{\partial(\bar{I}_1-3)}{\partial I_1}\mathbf{I}+\frac{\partial(\bar{I}_1-3)}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right]+2\gamma\left[\frac{\partial(\bar{I}_4-1)}{\partial I_4}\mathbf{M}+\frac{\partial(\bar{I}_4-1)}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right]\right)$$

$$-\frac{1}{3}\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[\frac{\partial \bar{I}_4}{\partial I_4}\mathbf{M}+\frac{\partial \bar{I}_4}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right]+\bar{I}_4\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}}:\frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}\right)$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
=&\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\beta\left[J^{-\frac{2}{3}}\mathbf{I}-\frac{2}{3}J^{-\frac{5}{3}}I_1\frac{J}{2}\mathbf{C}^{-1}\right]+2\gamma\left[J^{-\frac{2}{3}}\mathbf{M}-\frac{2}{3}J^{-\frac{5}{3}}I_4\frac{J}{2}\mathbf{C}^{-1}\right]\right)\\
&-\frac{1}{3}\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[J^{-\frac{2}{3}}\mathbf{M}-\frac{2}{3}J^{-\frac{5}{3}}I_4\frac{J}{2}\mathbf{C}^{-1}\right]+J^{-\frac{2}{3}}\bar{I}_4\frac{\partial\bar{\mathbf{C}}^{-1}}{\partial\bar{\mathbf{C}}}:\left[\mathbb{I}^{sym}-\frac{1}{3}\bar{\mathbf{C}}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
=&J^{-\frac{2}{3}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\beta\left[\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right]+2\gamma\left[\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right)\\
&-\frac{1}{3}J^{-\frac{2}{3}}\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]-\bar{I}_4\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
\tag{43}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$\boxed{\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)}$$

$$= (\mathbf{M} \boxtimes \mathbf{I}) : \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}} + (\mathbf{I} \boxtimes \bar{\mathbf{C}}) : \frac{\partial \mathbf{M}}{\partial \mathbf{C}} + (\bar{\mathbf{C}} \boxtimes \mathbf{I}) : \frac{\partial \mathbf{M}}{\partial \mathbf{C}} + (\mathbf{I} \boxtimes \mathbf{M}) : \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}} - \frac{2}{3}\left(\bar{\mathbf{C}}^{-1} \otimes \frac{\partial \bar{I}_5}{\partial \mathbf{C}} + \bar{I}_5 \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \mathbf{C}}\right)$$

$$= J^{-\frac{2}{3}} (\mathbf{M} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{M}) : \left(\mathbb{I}^{sym} - \frac{1}{3}\bar{\mathbf{C}} \otimes \bar{\mathbf{C}}^{-1}\right) - \frac{2}{3}\left(\bar{\mathbf{C}}^{-1} \otimes \left[\frac{\partial \bar{I}_5}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial \bar{I}_5}{\partial I_5}(\mathbf{MC} + \mathbf{CM})\right] + \bar{I}_5 \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}} : \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}\right)$$

$$= J^{-\frac{2}{3}} (\mathbf{M} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{M}) : \left(\mathbb{I}^{sym} - \frac{1}{3}\bar{\mathbf{C}} \otimes \bar{\mathbf{C}}^{-1}\right) - \frac{2}{3}\left(\bar{\mathbf{C}}^{-1} \otimes \left[-\frac{4}{3}J^{-\frac{7}{3}} I_5 \frac{J}{2}\mathbf{C}^{-1} + J^{-\frac{4}{3}}(\mathbf{MC} + \mathbf{CM})\right]\right.$$

$$\left. + J^{-\frac{2}{3}}\bar{I}_5 \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}} : \left[\mathbb{I}^{sym} - \frac{1}{3}\bar{\mathbf{C}} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \tag{44}$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
=& J^{-\frac{2}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\left(\mathbb{I}^{sym}-\frac{1}{3}\mathbf{C}\otimes\mathbf{C}^{-1}\right)-\frac{2}{3}\left(\bar{\mathbf{C}}^{-1}\otimes\left[-\frac{2}{3}J^{-\frac{2}{3}}\bar{I}_5\bar{\mathbf{C}}^{-1}+J^{-\frac{2}{3}}(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M})\right]\right.\\
&\left.-J^{-\frac{2}{3}}\bar{I}_5\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
=& J^{-\frac{2}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\left(\mathbb{I}^{sym}-\frac{1}{3}\bar{\mathbf{C}}\otimes\bar{\mathbf{C}}^{-1}\right)-\frac{2}{3}J^{-\frac{2}{3}}\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}-\frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right]\right.\\
&\left.-\bar{I}_5\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\\
=& J^{-\frac{2}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\mathbb{I}^{sym}-\frac{1}{3}J^{-\frac{2}{3}}\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1}-\frac{2}{3}J^{-\frac{2}{3}}\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}-\frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right]\right.\\
&\left.-\bar{I}_5\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
\tag{45}
$$

Anisotropic Material Elasticity Tensor (PK2)

Inserting these terms back in Eqn. 40, the anisotropic material elasticity tensor $\mathbb{H}^{\mathbf{S}_{ani}}$ finally writes

$$
\begin{aligned}
\mathbb{H}^{\mathbf{S}_{ani}} =& 4 \Bigg[\beta(\bar{I}_4 - 1) \left(\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1} \right) + \left(\alpha + \beta \left[\bar{I}_1 - 3\right] + 2\gamma \left[\bar{I}_4 - 1\right]\right) \left(\mathbf{M} - \frac{1}{3}\bar{I}_4 \bar{\mathbf{C}}^{-1} \right) \\
& - \frac{\alpha}{2} \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5 \bar{\mathbf{C}}^{-1} \right) \Bigg] \otimes \left(-\frac{1}{3} J^{-\frac{4}{3}} \bar{\mathbf{C}}^{-1} \right) + 4J^{-\frac{2}{3}} \Bigg[\beta J^{-\frac{2}{3}} \left(\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1} \right) \otimes \left(\mathbf{M} - \frac{1}{3}\bar{I}_4 \bar{\mathbf{C}}^{-1} \right) \\
& - \frac{1}{3}\beta J^{-\frac{2}{3}} \left(\bar{I}_4 - 1\right) \left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1} \right] - \bar{I}_1 \left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} \right] \right) \Bigg] \\
& + 4J^{-\frac{2}{3}} \Bigg[J^{-\frac{2}{3}} \left(\mathbf{M} - \frac{1}{3}\bar{I}_4 \bar{\mathbf{C}}^{-1} \right) \otimes \left(\beta \left[\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1} \right] + 2\gamma \left[\mathbf{M} - \frac{1}{3}\bar{I}_4 \bar{\mathbf{C}}^{-1} \right] \right) \\
& - \frac{1}{3} J^{-\frac{2}{3}} \left(\alpha + \beta \left[\bar{I}_1 - 3\right] + 2\gamma \left[\bar{I}_4 - 1\right] \right) \left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M} - \frac{1}{3}\bar{I}_4 \bar{\mathbf{C}}^{-1} \right] - \bar{I}_4 \left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} \right] \right) \Bigg] \\
& - 2\alpha J^{-\frac{2}{3}} \Bigg[J^{-\frac{2}{3}} \left(\mathbf{M} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{M} \right) : \mathbb{I}^{sym} - \frac{1}{3} J^{-\frac{2}{3}} \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} \right) \otimes \bar{\mathbf{C}}^{-1} - \frac{2}{3} J^{-\frac{2}{3}} \Bigg(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5 \bar{\mathbf{C}}^{-1} \right] \\
& - \bar{I}_5 \left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} \right] \Bigg) \Bigg]
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
= &-\frac{4}{3}\beta J^{-\frac{4}{3}}(\bar{I}_4 - 1)\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} - \frac{4}{3}J^{-\frac{4}{3}}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} \\
&+ \frac{2}{3}\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} + 4\beta J^{-\frac{4}{3}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) \\
&- \frac{4}{3}\beta J^{-\frac{4}{3}}\left(\bar{I}_4 - 1\right)\left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right] - \bar{I}_1\left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \\
&+ 4J^{-\frac{4}{3}}\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) \otimes \left(\beta\left[\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right] + 2\gamma\left[\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right) \\
&- \frac{4}{3}J^{-\frac{4}{3}}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right] - \bar{I}_4\left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \\
&- 2\alpha J^{-\frac{4}{3}}\left(\mathbf{M} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{M}\right) : \mathbb{I}^{sym} + \frac{2}{3}\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right) \otimes \bar{\mathbf{C}}^{-1} \\
&+ \frac{4}{3}\alpha J^{-\frac{4}{3}}\left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right] - \bar{I}_5\left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
= &-\frac{4}{3}\beta J^{-\frac{4}{3}}\left(\bar{I}_4 - 1\right)\left[\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) - \bar{I}_1\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right] \\
&-\frac{4}{3}J^{-\frac{4}{3}}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left[\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right. \\
&\left.-\bar{I}_4\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right] + 8\gamma J^{-\frac{4}{3}}\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) - 2\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{M}\right):\mathbb{I}^{sym} \\
&+4\beta J^{-\frac{4}{3}}\left[\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) + \left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\right] \\
&+\frac{2}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right] \\
&+\frac{2}{3}\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1} + \frac{2}{3}\alpha J^{-\frac{4}{3}}\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right] - 2\bar{I}_5\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
= & -\frac{4}{3}\beta J^{-\frac{4}{3}}\left(\bar{I}_4-1\right)\left[\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)-\bar{I}_1\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right] \\
& -\frac{4}{3}J^{-\frac{4}{3}}\left(\alpha+\beta\left[\bar{I}_1-3\right]+2\gamma\left[\bar{I}_4-1\right]\right)\left[\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right. \\
& \left.-\bar{I}_4\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right]+8\gamma J^{-\frac{4}{3}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)-2\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\mathbb{I}^{sym} \\
& +4\beta J^{-\frac{4}{3}}\left[\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)+\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\right] \\
& +\frac{2}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}-\frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}-\frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right] \\
& +\frac{2}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)\right]-\frac{4}{3}\alpha\bar{I}_5J^{-\frac{4}{3}}\left(\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}+\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
&= -\frac{4}{3}\beta J^{-\frac{4}{3}}\left(\bar{I}_4 - 1\right)\left[\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) - \bar{I}_1\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right] \\
&\quad - \frac{4}{3}J^{-\frac{4}{3}}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left[\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right. \\
&\quad \left. -\bar{I}_4\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right] + 8\gamma J^{-\frac{4}{3}}\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) - 2\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{M}\right) : \mathbb{I}^{sym} \\
&\quad + 4\beta J^{-\frac{4}{3}}\left[\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) + \left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\right] \\
&\quad + \frac{2}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right] \\
&\quad + \frac{2}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right)\right] - \frac{4}{3}\alpha\bar{I}_5 J^{-\frac{4}{3}}\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}
\end{aligned} \tag{46}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

The anisotropic material elasticity tensor with respect to Cauchy stresses $\mathbb{H}^{\boldsymbol{\sigma}_{ani}}$ is obtained from the push-forward operation of Eqn. 40 to the current configuration

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}_{ani}} =& \frac{1}{J}\mathbf{F}\left(\mathbf{F}\mathbb{H}^{\mathbf{S}_{ani}}\mathbf{F}^T\right)\mathbf{F}^T = \frac{1}{J}J^{-\frac{4}{3}}\bar{\mathbf{F}}\left(\bar{\mathbf{F}}\mathbb{H}^{\mathbf{S}_{ani}}\bar{\mathbf{F}}^T\right)\bar{\mathbf{F}}^T \\
=& -\frac{4}{3J}\beta\left(\bar{I}_4 - 1\right)\left[\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right) - \bar{I}_1\left(\mathbf{I}\odot\mathbf{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right] \\
& - \frac{4}{3J}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left[\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right) - \bar{I}_4\left(\mathbf{I}\odot\mathbf{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right] \\
& + \frac{8}{J}\gamma\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\otimes\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right) - \frac{2}{J}\alpha\left(\bar{\mathbf{N}}\boxtimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\boxtimes\bar{\mathbf{N}}\right) : \mathbb{I}^{sym} \\
& + \frac{4}{J}\beta\left[\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right)\otimes\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right) + \left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\otimes\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right)\right] \\
& + \frac{2}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}} - \frac{2}{3}\bar{I}_5\mathbf{I}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}} - \frac{2}{3}\bar{I}_5\mathbf{I}\right)\right] \\
& + \frac{2}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right] - \frac{4}{3J}\alpha\bar{I}_5\mathbf{I}\odot\mathbf{I}
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

$$
\begin{aligned}
= &-\frac{4}{3J}\beta\left(\bar{I}_4 - 1\right)\left[\operatorname{dev}\left(\bar{\mathbf{B}}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\operatorname{dev}\left(\bar{\mathbf{B}}\right) - \bar{I}_1\left(\mathbb{I}^{sym} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right] \\
&-\frac{4}{3J}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left[\operatorname{dev}\left(\bar{\mathbf{N}}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\operatorname{dev}\left(\bar{\mathbf{N}}\right) - \bar{I}_4\left(\mathbb{I}^{sym} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right] \\
&+\frac{8}{J}\gamma\operatorname{dev}\left(\bar{\mathbf{N}}\right)\otimes\operatorname{dev}\left(\bar{\mathbf{N}}\right) - \frac{2}{J}\alpha\left(\bar{\mathbf{N}}\boxtimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\boxtimes\bar{\mathbf{N}}\right):\mathbb{I}^{sym} \\
&+\frac{4}{J}\beta\left[\operatorname{dev}\left(\bar{\mathbf{B}}\right)\otimes\operatorname{dev}\left(\bar{\mathbf{N}}\right) + \operatorname{dev}\left(\bar{\mathbf{N}}\right)\otimes\operatorname{dev}\left(\bar{\mathbf{B}}\right)\right] \\
&+\frac{2}{3J}\alpha\left[\left(\operatorname{dev}\left(\bar{\mathbf{N}}\bar{\mathbf{B}}\right) + \operatorname{dev}\left(\bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\operatorname{dev}\left(\bar{\mathbf{N}}\bar{\mathbf{B}}\right) + \operatorname{dev}\left(\bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right)\right] \\
&+\frac{2}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right] - \frac{4}{3J}\alpha\bar{I}_5\mathbb{I}^{sym}
\end{aligned} \tag{47}
$$

This expression presents a major and a minor symmetry as expected.

Anisotropic Material Elasticity Tensor (Cauchy)

Alternatively, Eqn. 47 can be re-written in a more compact form as

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}_{ani}} = &- \frac{4}{3J}\beta\left(\bar{I}_4 - 1\right)\left[\bar{\mathbf{B}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{B}} - \bar{I}_1\left(\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right)\right] \\
&- \frac{4}{3J}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left[\bar{\mathbf{N}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{N}} - \bar{I}_4\left(\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right)\right] \\
&+ \frac{4}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right) \otimes \mathbf{I} + \mathbf{I} \otimes \left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right) - \bar{I}_5\left(\mathbb{I}^{sym} + \frac{2}{3}\mathbf{I} \otimes \mathbf{I}\right)\right] + \frac{8}{J}\gamma \operatorname{dev}\left(\bar{\mathbf{N}}\right) \otimes \operatorname{dev}\left(\bar{\mathbf{N}}\right) \\
&+ \frac{4}{J}\beta\left[\operatorname{dev}\left(\bar{\mathbf{B}}\right) \otimes \operatorname{dev}\left(\bar{\mathbf{N}}\right) + \operatorname{dev}\left(\bar{\mathbf{N}}\right) \otimes \operatorname{dev}\left(\bar{\mathbf{B}}\right)\right] - \frac{2}{J}\alpha\left(\bar{\mathbf{N}} \boxtimes \bar{\mathbf{B}} + \bar{\mathbf{B}} \boxtimes \bar{\mathbf{N}}\right) : \mathbb{I}^{sym}
\end{aligned}
\tag{48}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

Using index notation, Eqn. 48 finally writes

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}\,ani}_{ijkl} = &- \frac{4}{3J}\beta\left(\bar{I}_4 - 1\right)\left[\bar{B}_{ij}\delta_{kl} + \delta_{ij}\bar{B}_{kl} - \bar{I}_1\left(\frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] + \frac{1}{3}\delta_{ij}\delta_{kl}\right)\right] \\
&- \frac{4}{3J}\left(\alpha + \beta\left[\bar{I}_1 - 3\right] + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left[\bar{N}_{ij}\delta_{kl} + \delta_{ij}\bar{N}_{kl} - \bar{I}_4\left(\frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] + \frac{1}{3}\delta_{ij}\delta_{kl}\right)\right] \\
&+ \frac{4}{3J}\alpha\left[\left(\bar{N}_{in}\bar{B}_{nj} + \bar{B}_{in}\bar{N}_{nj}\right)\delta_{kl} + \delta_{ij}\left(\bar{N}_{kn}\bar{B}_{nl} + \bar{B}_{kn}\bar{N}_{nl}\right) - \bar{I}_5\left(\frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] + \frac{2}{3}\delta_{ij}\delta_{kl}\right)\right] \\
&+ \frac{8}{J}\gamma\left(\bar{N}_{ij} - \frac{1}{3}\bar{N}_{nn}\delta_{ij}\right)\left(\bar{N}_{kl} - \frac{1}{3}\bar{N}_{nn}\delta_{kl}\right) \\
&+ \frac{4}{J}\beta\left[\left(\bar{B}_{ij} - \frac{1}{3}\bar{B}_{nn}\delta_{ij}\right)\left(\bar{N}_{kl} - \frac{1}{3}\bar{N}_{nn}\delta_{kl}\right) + \left(\bar{N}_{ij} - \frac{1}{3}\bar{N}_{nn}\delta_{ij}\right)\left(\bar{B}_{kl} - \frac{1}{3}\bar{B}_{nn}\delta_{kl}\right)\right] \\
&- \frac{1}{J}\alpha\left(\bar{N}_{ik}\bar{B}_{jl} + \bar{N}_{il}\bar{B}_{jk} + \bar{B}_{ik}\bar{N}_{jl} + \bar{B}_{il}\bar{N}_{jk}\right)
\end{aligned}
\tag{49}
$$

Table of Contents

Alternative Strain-Energy Density Function

In their work, [Bonet and Burton, 1998] have expressed the following concerns with the transversely isotropic component of the constitutive model (Eqn. 32):

"The transversely isotropic component is still that of the St.Venant model and is therefore linear with respect to the Lagrangian strain tensor. This could still cause difficulties in large strain application (...) In addition, (...) the resulting elasticity tensor in the deformed setting is not transversely isotropic."

Alternative Strain-Energy Density Function

To overcome these two concerns, they propose to replace the transversely isotropic component $(\bar{I}_1 - 3)$ of the strain energy function ψ_a with a term which is linear in $\mathbf{C}$

$$\psi_a = \left[\alpha + \beta \ln J + \gamma \left(\bar{I}_4 - 1\right)\right] \left(\bar{I}_4 - 1\right) - \frac{1}{2}\alpha(\bar{I}_5 - 1) \tag{50}$$

which has been here written using the reduced invariants.

The anisotropic PK2 stress tensor $\mathbf{S}_{ani}$ is obtained from the differentiation of ψ_a with respect to $\bar{\mathbf{C}}$ as in Eqn. 11

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}} \left[\frac{\partial \bar{\psi}}{\partial J}\frac{J}{2}\bar{\mathbf{C}}^{-1} + \frac{\partial \bar{\psi}}{\partial \bar{I}_4}\mathbf{M} + \frac{\partial \bar{\psi}}{\partial \bar{I}_5}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right)\right] \tag{51}$$

Anisotropic PK2 Stress Tensor

Managing each differential term separately

$$\frac{\partial \psi_a}{\partial \bar{I}_4} = \alpha + \beta \ln J + 2\gamma(\bar{I}_4 - 1) \tag{52}$$

$$\frac{\partial \psi_a}{\partial \bar{I}_5} = -\frac{\alpha}{2} \tag{53}$$

$$\begin{aligned}\frac{\partial \psi_a}{\partial J} &= -\frac{2}{3}J^{-\frac{5}{3}} I_4 \left[\alpha + \beta \ln J + \gamma \left(\bar{I}_4 - 1\right)\right] + \left[\beta \frac{1}{J} - \frac{2}{3}\gamma J^{-\frac{5}{3}} I_4\right] \left(\bar{I}_4 - 1\right) + \frac{4}{3}\frac{\alpha}{2} J^{-\frac{7}{3}} I_5 \\ &= -\frac{2}{3}\frac{1}{J}\bar{I}_4 \left[\alpha + \beta \ln J + \gamma \left(\bar{I}_4 - 1\right)\right] + \frac{1}{J}\left[\beta - \frac{2}{3}\gamma \bar{I}_4\right] \left(\bar{I}_4 - 1\right) + \frac{2}{3}\frac{\alpha}{J}\bar{I}_5\end{aligned} \tag{54}$$

Anisotropic PK2 Stress Tensor

and injecting those terms in Eqn. 51

$$
\begin{aligned}
\mathbf{S}_{ani} =& 2J^{-\frac{2}{3}} \left[\left(-\frac{2}{3}\frac{1}{J}\bar{I}_4 \left[\alpha + \beta \ln J + \gamma \left(\bar{I}_4 - 1\right)\right] + \frac{1}{J}\left[\beta - \frac{2}{3}\gamma\bar{I}_4\right]\left(\bar{I}_4 - 1\right) + \frac{2}{3}\frac{\alpha}{J}\bar{I}_5 \right) \frac{J}{2}\bar{\mathbf{C}}^{-1} \right. \\
& \left. + \left(\alpha + \beta \ln J + 2\gamma \left[\bar{I}_4 - 1\right]\right)\mathbf{M} - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right) \right] \\
=& 2J^{-\frac{2}{3}} \left[\left(-\frac{1}{3}\bar{I}_4 \left[\alpha + \beta \ln J + \gamma \left(\bar{I}_4 - 1\right)\right] + \left[\frac{1}{2}\beta - \frac{1}{3}\gamma\bar{I}_4\right]\left(\bar{I}_4 - 1\right) + \frac{1}{3}\alpha\bar{I}_5 \right) \bar{\mathbf{C}}^{-1} \right. \\
& \left. + \left(\alpha + \beta \ln J + 2\gamma \left[\bar{I}_4 - 1\right]\right)\mathbf{M} - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right) \right] \\
=& 2J^{-\frac{2}{3}} \left[\frac{1}{2}\beta\left(\bar{I}_4 - 1\right)\bar{\mathbf{C}}^{-1} + \left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) \right. \\
& \left. - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right) \right]
\end{aligned}
\tag{55}
$$

Anisotropic Cauchy Stress Tensor

The anisotropic Cauchy stress tensor $\boldsymbol{\sigma}_{ani}$ is obtained from the push-forward of Eqn. 55

$$
\begin{aligned}
\boldsymbol{\sigma}_{ani} =& \frac{1}{J}\mathbf{F}\mathbf{S}_{ani}\mathbf{F}^T = \frac{1}{J}J^{\frac{2}{3}}\bar{\mathbf{F}}\mathbf{S}_{ani}\bar{\mathbf{F}}^T \\
=& \frac{2}{J}\Bigg[\frac{1}{2}\beta\left(\bar{I}_4 - 1\right)\bar{\mathbf{F}}\bar{\mathbf{C}}^{-1}\bar{\mathbf{F}}^T + \left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\bar{\mathbf{F}}\mathbf{M}\bar{\mathbf{F}}^T - \frac{1}{3}\bar{I}_4\bar{\mathbf{F}}\bar{\mathbf{C}}^{-1}\bar{\mathbf{F}}^T\right) \\
& - \frac{\alpha}{2}\left(\bar{\mathbf{F}}\mathbf{M}\bar{\mathbf{C}}\bar{\mathbf{F}}^T + \bar{\mathbf{F}}\bar{\mathbf{C}}\mathbf{M}\bar{\mathbf{F}}^T - \frac{2}{3}\bar{I}_5\bar{\mathbf{F}}\bar{\mathbf{C}}^{-1}\bar{\mathbf{F}}^T\right)\Bigg] \\
=& \frac{2}{J}\left[\frac{1}{2}\beta\left(\bar{I}_4 - 1\right)\mathbf{I} + \left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right) - \frac{\alpha}{2}\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}} - \frac{2}{3}\bar{I}_5\mathbf{I}\right)\right] \\
=& \frac{\beta}{J}\left(\bar{I}_4 - 1\right)\mathbf{I} + \frac{2}{J}\left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\operatorname{dev}\left(\bar{\mathbf{N}}\right) - \frac{\alpha}{J}\left(\operatorname{dev}\left(\bar{\mathbf{N}}\bar{\mathbf{B}}\right) + \operatorname{dev}\left(\bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right) \qquad (56)
\end{aligned}
$$

Anisotropic Deviatoric Stress and Pressure

Eqn. 56 now contributes both to the deviatoric stress $\mathbf{s}$

$$\mathbf{s}_{ani} = \text{dev}\left(\boldsymbol{\sigma}_{ani}\right) = \frac{2}{J}\left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right) \text{dev}\left(\bar{\mathbf{N}}\right) - \frac{\alpha}{J}\left(\text{dev}\left(\bar{\mathbf{N}}\bar{\mathbf{B}}\right) + \text{dev}\left(\bar{\mathbf{B}}\bar{\mathbf{N}}\right)\right) \tag{57}$$

and to the anisotropic pressure contribution p_{ani}

$$p_{ani} = \frac{\text{tr}\left(\boldsymbol{\sigma}_{ani}\right)}{3} = \frac{\beta}{J}\left(\bar{I}_4 - 1\right) \tag{58}$$

Anisotropic Material Elasticity Tensor (PK2)

The material elasticity tensor $\mathbb{H}^{\mathbf{S}_{ani}}$ is obtained from the differentiation of Eqn. 55

$$
\begin{aligned}
\mathbb{H}^{\mathbf{S}_{ani}} = 2\frac{\partial \mathbf{S}_{ani}}{\partial \mathbf{C}} =& 2\frac{\partial}{\partial \mathbf{C}}\left(2J^{-\frac{2}{3}}\left[\frac{\beta}{2}\left(\bar{I}_4 - 1\right)\bar{\mathbf{C}}^{-1} + \left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right.\right.\\
&\left.\left. - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right]\right)\\
=& 4\left[\frac{\beta}{2}\left(\bar{I}_4 - 1\right)\bar{\mathbf{C}}^{-1} + \left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right.\\
&\left. - \frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\right] \otimes \frac{\partial}{\partial \mathbf{C}}\left(J^{-\frac{2}{3}}\right) + 2J^{-\frac{2}{3}}\beta\frac{\partial}{\partial \mathbf{C}}\left(\left[\bar{I}_4 - 1\right]\bar{\mathbf{C}}^{-1}\right)\\
&+ 4J^{-\frac{2}{3}}\frac{\partial}{\partial \mathbf{C}}\left(\left[\alpha + \beta \ln J + 2\gamma\left(\bar{I}_4 - 1\right)\right]\left[\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right) - 2\alpha J^{-\frac{2}{3}}\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)
\end{aligned}
\tag{59}
$$

For clarity and conciseness, each terms from Eqn. 59 are derived separately hereafter (except for terms that have already been derived in Eqn. 41 and Eqn. 44).

Anisotropic Material Elasticity Tensor (PK2)

$$\boxed{\frac{\partial}{\partial \mathbf{C}}\left(\left[\bar{I}_4 - 1\right]\,\bar{\mathbf{C}}^{-1}\right)}$$

$$\begin{aligned}
&=\bar{\mathbf{C}}^{-1} \otimes \frac{\partial}{\partial \mathbf{C}}\left(\left[\bar{I}_4 - 1\right]\right) + \left(\bar{I}_4 - 1\right)\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \mathbf{C}} \\
&=\bar{\mathbf{C}}^{-1} \otimes \left(\frac{\partial\left(\bar{I}_4 - 1\right)}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial\left(\bar{I}_4 - 1\right)}{\partial I_4}\mathbf{M}\right) - J^{-\frac{2}{3}}\left(\bar{I}_4 - 1\right)\left(\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right) \\
&=\bar{\mathbf{C}}^{-1} \otimes \left(-\frac{2}{3}J^{-\frac{5}{3}} I_4 \frac{J}{2}\mathbf{C}^{-1} + J^{-\frac{2}{3}}\mathbf{M}\right) - J^{-\frac{2}{3}}\left(\bar{I}_4 - 1\right)\left(\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right) \\
&=J^{-\frac{2}{3}}\left[\bar{\mathbf{C}}^{-1} \otimes \left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) - \left(\bar{I}_4 - 1\right)\left(\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right)\right] \qquad (60)
\end{aligned}$$

Anisotropic Material Elasticity Tensor (PK2)

$$\boxed{\frac{\partial}{\partial \mathbf{C}}\left(\left[\alpha+\beta\ln J+2\gamma\left(\bar{I}_4-1\right)\right]\left[\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right)}$$

$$=\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\frac{\partial}{\partial \mathbf{C}}\left(\alpha+\beta\ln J+2\gamma\left[\bar{I}_4-1\right]\right)+\left(\alpha+\beta\ln J+2\gamma\left[\bar{I}_4-1\right]\right)\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)$$

$$=\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\beta\frac{\partial\left(\ln J\right)}{\partial \mathbf{C}}+2\gamma\frac{\partial\left(\bar{I}_4-1\right)}{\partial \mathbf{C}}\right)$$

$$+\left(\alpha+\beta\ln J+2\gamma\left[\bar{I}_4-1\right]\right)\left(\frac{\partial \mathbf{M}}{\partial \mathbf{C}}-\frac{1}{3}\left[\bar{\mathbf{C}}^{-1}\otimes\frac{\partial \bar{I}_4}{\partial \mathbf{C}}+\bar{I}_4\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \mathbf{C}}\right]\right)$$

$$=\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\beta\frac{\partial\left(\ln J\right)}{\partial J}\frac{J}{2}\mathbf{C}^{-1}+2\gamma\left[\frac{\partial(\bar{I}_4-1)}{\partial I_4}\mathbf{M}+\frac{\partial(\bar{I}_4-1)}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right]\right)$$

$$-\frac{1}{3}\left(\alpha+\beta\ln J+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[\frac{\partial \bar{I}_4}{\partial I_4}\mathbf{M}+\frac{\partial \bar{I}_4}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right]+\bar{I}_4\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}}:\frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}\right)$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
&= \left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) \otimes \left(\beta \frac{1}{J}\frac{J}{2}\mathbf{C}^{-1} + 2\gamma\left[J^{-\frac{2}{3}}\mathbf{M} - \frac{2}{3}J^{-\frac{5}{3}}I_4\frac{J}{2}\mathbf{C}^{-1}\right]\right) \\
&\quad - \frac{1}{3}\left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\bar{\mathbf{C}}^{-1} \otimes \left[J^{-\frac{2}{3}}\mathbf{M} - \frac{2}{3}J^{-\frac{5}{3}}I_4\frac{J}{2}\mathbf{C}^{-1}\right] + J^{-\frac{2}{3}}\bar{I}_4\frac{\partial\bar{\mathbf{C}}^{-1}}{\partial\bar{\mathbf{C}}} : \left[\mathbb{I}^{sym} - \frac{1}{3}\bar{\mathbf{C}} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \\
&= J^{-\frac{2}{3}}\left(\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right) \otimes \left(\frac{\beta}{2}\bar{\mathbf{C}}^{-1} + 2\gamma\left[\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right) \\
&\quad - \frac{1}{3}J^{-\frac{2}{3}}\left(\alpha + \beta \ln J + 2\gamma\left[\bar{I}_4 - 1\right]\right)\left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M} - \frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right] - \bar{I}_4\left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \qquad (61)
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

Inserting the terms back in Eqn. 59, the anisotropic material elasticity tensor $\mathbb{H}^{\mathbf{S}_{ani}}$ finally writes

$$
\begin{aligned}
\mathbb{H}^{\mathbf{S}_{ani}} =& 4\Bigg[\frac{\beta}{2}\left(\bar{I}_4-1\right)\bar{\mathbf{C}}^{-1}+\left(\alpha+\beta\ln J+2\gamma\left[\bar{I}_4-1\right]\right)\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\\
&-\frac{\alpha}{2}\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}-\frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right)\Bigg]\otimes\left(-\frac{1}{3}J^{-\frac{4}{3}}\bar{\mathbf{C}}^{-1}\right)+2J^{-\frac{2}{3}}\beta\Bigg(J^{-\frac{2}{3}}\Bigg[\frac{1}{2}\left(\bar{I}_4-1\right)\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\\
&+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)-\left(\bar{I}_4-1\right)\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\Bigg]\Bigg)\\
&+4J^{-\frac{2}{3}}\Bigg[J^{-\frac{2}{3}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\frac{\beta}{2}\bar{\mathbf{C}}^{-1}+2\gamma\left[\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]\right)\\
&-\frac{1}{3}J^{-\frac{2}{3}}\left(\alpha+\beta\ln J+2\gamma\left[\bar{I}_4-1\right]\right)\left(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right]-\bar{I}_4\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\right)\Bigg]\\
&-2\alpha J^{-\frac{2}{3}}\Bigg[J^{-\frac{2}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\mathbb{I}^{sym}-\frac{1}{3}J^{-\frac{2}{3}}\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1}-\frac{2}{3}J^{-\frac{2}{3}}\Bigg(\bar{\mathbf{C}}^{-1}\otimes\left[\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}-\frac{2}{3}\bar{I}_5\bar{\mathbf{C}}^{-1}\right]\\
&-\bar{I}_5\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\Bigg)\Bigg]
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
= &- \frac{2}{3} J^{-\frac{4}{3}} \beta \left(\bar{I}_4 - 1\right) \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} - \frac{4}{3} J^{-\frac{4}{3}} \left(\alpha + \beta \ln J + 2\gamma \left[\bar{I}_4 - 1\right]\right) \left(\mathbf{M} - \frac{1}{3} \bar{I}_4 \bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} \\
&+ \frac{2}{3} J^{-\frac{4}{3}} \alpha \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3} \bar{I}_5 \bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} + J^{-\frac{4}{3}} \beta \left(\bar{I}_4 - 1\right) \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} \\
&+ 2 J^{-\frac{4}{3}} \beta \bar{\mathbf{C}}^{-1} \otimes \left(\mathbf{M} - \frac{1}{3} \bar{I}_4 \bar{\mathbf{C}}^{-1}\right) - 2 J^{-\frac{4}{3}} \beta \left(\bar{I}_4 - 1\right) \left(\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right) \\
&+ 2 J^{-\frac{4}{3}} \beta \left(\mathbf{M} - \frac{1}{3} \bar{I}_4 \bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} + 8 J^{-\frac{4}{3}} \gamma \left(\mathbf{M} - \frac{1}{3} \bar{I}_4 \bar{\mathbf{C}}^{-1}\right) \otimes \left(\mathbf{M} - \frac{1}{3} \bar{I}_4 \bar{\mathbf{C}}^{-1}\right) \\
&- \frac{4}{3} J^{-\frac{4}{3}} \left(\alpha + \beta \ln J + 2\gamma \left[\bar{I}_4 - 1\right]\right) \left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M} - \frac{1}{3} \bar{I}_4 \bar{\mathbf{C}}^{-1}\right] - \bar{I}_4 \left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \\
&- 2 J^{-\frac{4}{3}} \alpha \left(\mathbf{M} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{M}\right) : \mathbb{I}^{sym} + \frac{2}{3} J^{-\frac{4}{3}} \alpha \left(\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M}\right) \otimes \bar{\mathbf{C}}^{-1} \\
&+ \frac{4}{3} J^{-\frac{4}{3}} \alpha \left(\bar{\mathbf{C}}^{-1} \otimes \left[\mathbf{M}\bar{\mathbf{C}} + \bar{\mathbf{C}}\mathbf{M} - \frac{2}{3} \bar{I}_5 \bar{\mathbf{C}}^{-1}\right] - \bar{I}_5 \left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right)
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
=&2\beta J^{-\frac{4}{3}}\left[\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right]+8\gamma J^{-\frac{4}{3}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\\
&+J^{-\frac{4}{3}}\beta\left(\bar{I}_4-1\right)\left[\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}-\frac{2}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}+\frac{2}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}-2\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}\right]\\
&-\frac{4}{3}J^{-\frac{4}{3}}\left[\alpha+\beta\ln J+2\gamma\left(\bar{I}_4-1\right)\right]\left[\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)+\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}\right.\\
&\left.-\bar{I}_4\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right]-2\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\mathbb{I}^{sym}\\
&+\frac{4}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)-\bar{I}_5\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}+\frac{2}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right]
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (PK2)

$$
\begin{aligned}
=&2\beta J^{-\frac{4}{3}}\left[\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\right]+8\gamma J^{-\frac{4}{3}}\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\\
&-2J^{-\frac{4}{3}}\beta\left(\bar{I}_4-1\right)\left[\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{2}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right]\\
&-\frac{4}{3}J^{-\frac{4}{3}}\left[\alpha+\beta\ln J+2\gamma\left(\bar{I}_4-1\right)\right]\left[\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)+\left(\mathbf{M}-\frac{1}{3}\bar{I}_4\bar{\mathbf{C}}^{-1}\right)\otimes\bar{\mathbf{C}}^{-1}\right.\\
&\left.-\bar{I}_4\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}-\frac{1}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right]-2\alpha J^{-\frac{4}{3}}\left(\mathbf{M}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{M}\right):\mathbb{I}^{sym}\\
&+\frac{4}{3}\alpha J^{-\frac{4}{3}}\left[\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)\otimes\bar{\mathbf{C}}^{-1}+\bar{\mathbf{C}}^{-1}\otimes\left(\mathbf{M}\bar{\mathbf{C}}+\bar{\mathbf{C}}\mathbf{M}\right)-\bar{I}_5\left(\bar{\mathbf{C}}^{-1}\odot\bar{\mathbf{C}}^{-1}+\frac{2}{3}\bar{\mathbf{C}}^{-1}\otimes\bar{\mathbf{C}}^{-1}\right)\right]
\end{aligned}
\tag{62}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

The anisotropic material elasticity tensor with respect to Cauchy stresses $\mathbb{H}^{\boldsymbol{\sigma}_{ani}}$ is obtained from the push-forward operation of Eqn. 62 to the current configuration

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}_{ani}} =& \frac{1}{J}\mathbf{F}\left(\mathbf{F}\mathbb{H}^{\mathbf{S}_{ani}}\mathbf{F}^T\right)\mathbf{F}^T = \frac{1}{J}J^{-\frac{4}{3}}\bar{\mathbf{F}}\left(\bar{\mathbf{F}}\mathbb{H}^{\mathbf{S}_{ani}}\bar{\mathbf{F}}^T\right)\bar{\mathbf{F}}^T \\
=& 2\frac{\beta}{J}\left[\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\right] + \frac{8}{J}\gamma\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\otimes\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right) \\
& - \frac{\beta}{J}\left(\bar{I}_4 - 1\right)\left[2\mathbb{I}^{sym} - \mathbf{I}\otimes\mathbf{I}\right] \\
& - \frac{4}{3J}\left[\alpha + \beta\ln J + 2\gamma\left(\bar{I}_4 - 1\right)\right]\left[\mathbf{I}\otimes\left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right) + \left(\bar{\mathbf{N}} - \frac{1}{3}\bar{I}_4\mathbf{I}\right)\otimes\mathbf{I}\right. \\
& \left. - \bar{I}_4\left(\mathbb{I}^{sym} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right] - \frac{2}{J}\alpha\left(\bar{\mathbf{N}}\boxtimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\boxtimes\bar{\mathbf{N}}\right) : \mathbb{I}^{sym} \\
& + \frac{4}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right) - \bar{I}_5\left(\mathbb{I}^{sym} + \frac{2}{3}\mathbf{I}\otimes\mathbf{I}\right)\right]
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

$$
\begin{aligned}
=&2\frac{\beta}{J}\left[\operatorname{dev}\left(\bar{\mathbf{N}}\right)\otimes\mathbf{I}+\mathbf{I}\otimes\operatorname{dev}\left(\bar{\mathbf{N}}\right)\right]+\frac{8}{J}\gamma\operatorname{dev}\left(\bar{\mathbf{N}}\right)\otimes\operatorname{dev}\left(\bar{\mathbf{N}}\right)-\frac{2}{J}\alpha\left(\bar{\mathbf{N}}\boxtimes\bar{\mathbf{B}}+\bar{\mathbf{B}}\boxtimes\bar{\mathbf{N}}\right):\mathbb{I}^{sym}\\
&-\frac{\beta}{J}\left(\bar{I}_4-1\right)\left[2\mathbb{I}^{sym}-\mathbf{I}\otimes\mathbf{I}\right]\\
&-\frac{4}{3J}\left[\alpha+\beta\ln J+2\gamma\left(\bar{I}_4-1\right)\right]\left[\mathbf{I}\otimes\bar{\mathbf{N}}+\bar{\mathbf{N}}\otimes\mathbf{I}-\bar{I}_4\left(\mathbb{I}^{sym}+\frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right]\\
&+\frac{4}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}}+\bar{\mathbf{B}}\bar{\mathbf{N}}\right)\otimes\mathbf{I}+\mathbf{I}\otimes\left(\bar{\mathbf{N}}\bar{\mathbf{B}}+\bar{\mathbf{B}}\bar{\mathbf{N}}\right)-\bar{I}_5\left(\mathbb{I}^{sym}+\frac{2}{3}\mathbf{I}\otimes\mathbf{I}\right)\right]
\end{aligned}
\tag{63}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

Using index notation, Eqn. 63 finally writes

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}_{ani}}_{ijkl} =& 2\frac{\beta}{J}\left[\left(\bar{N}_{ij}-\frac{1}{3}\bar{N}_{nn}\delta_{ij}\right)\delta_{kl}+\delta_{ij}\left(\bar{N}_{kl}-\frac{1}{3}\bar{N}_{nn}\delta_{kl}\right)\right]+\frac{8}{J}\gamma\left(\bar{N}_{ij}-\frac{1}{3}\bar{N}_{nn}\delta_{ij}\right)\left(\bar{N}_{kl}-\frac{1}{3}\bar{N}_{nn}\delta_{kl}\right) \\
&-\frac{1}{J}\alpha\left(\bar{N}_{ik}\bar{B}_{jl}+\bar{N}_{il}\bar{B}_{jk}+\bar{B}_{ik}\bar{N}_{jl}+\bar{B}_{il}\bar{N}_{jk}\right)-\frac{\beta}{J}\left(\bar{I}_4-1\right)\left[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}\right] \\
&-\frac{4}{3J}\left[\alpha+\beta\ln J+2\gamma\left(\bar{I}_4-1\right)\right]\left[\bar{N}_{ij}\delta_{kl}+\delta_{ij}\bar{N}_{kl}-\bar{I}_4\left(\frac{1}{2}\left[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right]+\frac{1}{3}\delta_{ij}\delta_{kl}\right)\right] \\
&+\frac{4}{3J}\alpha\left[\left(\bar{N}_{in}\bar{B}_{nj}+\bar{B}_{in}\bar{N}_{nj}\right)\delta_{kl}+\delta_{ij}\left(\bar{N}_{kn}\bar{B}_{nl}+\bar{B}_{kn}\bar{N}_{nl}\right)-\bar{I}_5\left(\frac{1}{2}\left[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right]+\frac{2}{3}\delta_{ij}\delta_{kl}\right)\right] \qquad (64)
\end{aligned}
$$

Anisotropic Material Elasticity Tensor (Cauchy)

Taking into account the deviatoric/volumetric split in Metafor, the anisotropic material elasticity tensor can be separated into its deviatoric contribution $\mathbb{H}^{\boldsymbol{\sigma}_{ani}}_{dev}$

$$\begin{aligned}\mathbb{H}^{\boldsymbol{\sigma}_{ani}}_{dev} =& 2\frac{\beta}{J}\,\mathrm{dev}\left(\bar{\mathbf{N}}\right)\otimes\mathbf{I} + \frac{8}{J}\gamma\,\mathrm{dev}\left(\bar{\mathbf{N}}\right)\otimes\mathrm{dev}\left(\bar{\mathbf{N}}\right) - \frac{2}{J}\alpha\left(\bar{\mathbf{N}}\boxtimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\boxtimes\bar{\mathbf{N}}\right) : \mathbb{I}^{sym}\\ &- \frac{4}{3J}\left[\alpha + \beta\ln J + 2\gamma\left(\bar{I}_4 - 1\right)\right]\left[\mathbf{I}\otimes\bar{\mathbf{N}} + \bar{\mathbf{N}}\otimes\mathbf{I} - \bar{I}_4\left(\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)\right]\\ &+ \frac{4}{3J}\alpha\left[\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right)\otimes\mathbf{I} + \mathbf{I}\otimes\left(\bar{\mathbf{N}}\bar{\mathbf{B}} + \bar{\mathbf{B}}\bar{\mathbf{N}}\right) - \bar{I}_5\left(\mathbb{I}^{sym} + \frac{2}{3}\mathbf{I}\otimes\mathbf{I}\right)\right]\end{aligned} \tag{65}$$

and its volumetric contribution $\mathbb{H}^{\boldsymbol{\sigma}_{ani}}_{vol}$

$$\begin{aligned}\mathbb{H}^{\boldsymbol{\sigma}_{ani}}_{vol} =& 2\frac{\beta}{J}\,\mathbf{I}\otimes\mathrm{dev}\left(\bar{\mathbf{N}}\right) - \frac{\beta}{J}\left(\bar{I}_4 - 1\right)\left[2\mathbb{I}^{sym} - \mathbf{I}\otimes\mathbf{I}\right]\\ =& 2\frac{\beta}{J}\,\mathbf{I}\otimes\mathrm{dev}\left(\bar{\mathbf{N}}\right) + p_{ani}\left[\mathbf{I}\otimes\mathbf{I} - 2\mathbb{I}^{sym}\right]\end{aligned} \tag{66}$$

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Strain-Energy Density Function

The (general) strain-energy density function for a Yeoh material writes [4]

$$\psi_{Yeoh}(\bar{I}_1, J) = C_1(\bar{I}_1 - 3) + C_2(\bar{I}_1 - 3)^2 + C_3(\bar{I}_1 - 3)^3 + kf(J) = \psi^{dev}(\bar{I}_1) + \psi^{vol}(J) \tag{67}$$

In this case, we will focus solely on the deviatoric part ψ^{dev}, hence ommiting the $kf(J)$ term.

PK2 and Cauchy Stress Tensors

The deviatoric part of the PK2 stress tensor $\mathbf{S}_{iso}$ is obtained from the differentiation of Eqn. 67 with respect to $\bar{\mathbf{C}}$

$$\begin{aligned}\mathbf{S}^{dev} &= 2J^{-\frac{2}{3}} \frac{\partial \psi_{dev}}{\partial \bar{\mathbf{C}}} = 2J^{-\frac{2}{3}} \left[\frac{\partial \psi^{dev}}{\partial \bar{I}_1} \mathbf{I} + \frac{\partial \psi^{dev}}{\partial J} \frac{J}{2} \bar{\mathbf{C}}^{-1} \right] \\ &= 2J^{-\frac{2}{3}} \left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2 \right] \left(\mathbf{I} - \frac{1}{3} \bar{I}_1 \bar{\mathbf{C}}^{-1} \right) \end{aligned} \tag{68}$$

The deviatoric part of the Cauchy stress tensor $\boldsymbol{\sigma}$ is obtained from the push-forward of Eqn. 68

$$\boldsymbol{\sigma}^{dev} = \frac{1}{J} \mathbf{F} \mathbf{S}^{dev} \mathbf{F}^T = \frac{2}{J} \left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2 \right] \operatorname{dev}\left(\bar{\mathbf{B}} \right) \tag{69}$$

Material Elasticity Tensor (PK2)

The deviatoric part of the material elasticity tensor $\mathbb{H}^{\mathbf{S}}_{dev}$ writes

$$
\begin{aligned}
\mathbb{H}^{\mathbf{S}}_{dev} = 2\frac{\partial \mathbf{S}^{dev}}{\partial \mathbf{C}} =& 4\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \frac{\partial}{\partial \mathbf{C}}\left(J^{-\frac{2}{3}}\right) \\
&+ 4J^{-\frac{2}{3}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \frac{\partial}{\partial \mathbf{C}}\left(C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right) \\
&+ 4J^{-\frac{2}{3}}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \\
=& 4\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \left(-\frac{2}{3}J^{-\frac{5}{3}}\frac{J}{2}\mathbf{C}^{-1}\right) \\
&+ 4J^{-\frac{2}{3}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \left[\frac{\partial}{\partial I_1}\left(C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right)\mathbf{I}\right. \\
&\left.+\frac{\partial}{\partial J}\left(C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right)\frac{J}{2}\mathbf{C}^{-1}\right] \\
&- \frac{4}{3}J^{-\frac{2}{3}}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left[\bar{\mathbf{C}}^{-1} \otimes \left(\frac{\partial \bar{I}_1}{\partial I_1}\mathbf{I} + \frac{\partial \bar{I}_1}{\partial J}\frac{J}{2}\mathbf{C}^{-1}\right) + \bar{I}_1\frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \mathbf{C}}\right]
\end{aligned}
$$

Material Elasticity Tensor (PK2)

$$
\begin{aligned}
&= -\frac{4}{3}J^{-\frac{4}{3}}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \bar{\mathbf{C}}^{-1} \\
&\quad + 4J^{-\frac{4}{3}}\left[2C_2 + 6C_3(\bar{I}_1 - 3)\right]\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \\
&\quad - \frac{4}{3}J^{-\frac{4}{3}}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\bar{\mathbf{C}}^{-1} \otimes \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) - \bar{I}_1\left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} - \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \\
&= -\frac{4}{3}J^{-\frac{4}{3}}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\mathbf{I} \otimes \bar{\mathbf{C}}^{-1} + \bar{\mathbf{C}}^{-1} \otimes \mathbf{I} - \bar{I}_1\left[\bar{\mathbf{C}}^{-1} \odot \bar{\mathbf{C}}^{-1} + \frac{1}{3}\bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1}\right]\right) \\
&\quad + 4J^{-\frac{4}{3}}\left[2C_2 + 6C_3(\bar{I}_1 - 3)\right]\left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \otimes \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) \qquad (70)
\end{aligned}
$$

Material Elasticity Tensor (Cauchy)

The deviatoric part of the isotropic material elasticity tensor with respect to Cauchy stresses $\mathbb{H}^{\boldsymbol{\sigma}}_{dev}$ is obtained from the push-forward operation of Eqn. 70 to the current configuration

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}\,iso}_{dev} &= \frac{1}{J}\mathbf{F}\left(\mathbf{F}\mathbb{H}^{\mathbf{S}}_{dev}\mathbf{F}^T\right)\mathbf{F}^T = \frac{1}{J}J^{-\frac{4}{3}}\bar{\mathbf{F}}\left(\bar{\mathbf{F}}\mathbb{H}^{\mathbf{S}}_{dev}\bar{\mathbf{F}}^T\right)\bar{\mathbf{F}}^T \\
&= -\frac{4}{3J}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\mathbf{I}\otimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\otimes\mathbf{I} - \bar{I}_1\left[\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right]\right) \\
&\quad + \frac{4}{J}\left[2C_2 + 6C_3(\bar{I}_1 - 3)\right]\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right)\otimes\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right) \\
&= -\frac{4}{3J}\left[C_1 + 2C_2(\bar{I}_1 - 3) + 3C_3(\bar{I}_1 - 3)^2\right]\left(\mathbf{I}\otimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\otimes\mathbf{I} - \bar{I}_1\left[\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right]\right) \\
&\quad + \frac{4}{J}\left[2C_2 + 6C_3(\bar{I}_1 - 3)\right]\operatorname{dev}\left(\bar{\mathbf{B}}\right)\otimes\operatorname{dev}\left(\bar{\mathbf{B}}\right)
\end{aligned}
\tag{71}
$$

This expression presents a major and a minor symmetry as expected.

References

M. Abd Latif, D. Perić, and W. Dettmer.
Numerical modeling of transversely isotropic elastic material at small and finite strains.
volume 909, pages 142–146, 05 2007.

J. Bonet and A. Burton.
A simple orthotropic, transversely isotropic hyperelastic constitutive equation for large strain computations.
Computer Methods in Applied Mechanics and Engineering, 162(1):151–164, 1998.

P. J. Flory.
Thermodynamic relations for high elastic materials.
Trans. Faraday Soc., 57:829–838, 1961.

References *(ctd.)*

C. Renaud, J.-M. Cros, Z.-Q. Feng, and B. Yang.
The yeoh model applied to the modeling of large deformation contact/impact problems.
International Journal of Impact Engineering, 36(5):659–666, 2009.