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Aerospace & Mechanical engineering

Holzapfel-Gasser-Ogden (HGO) Anisotropic Hyperelastic Model Implementation in Metafor

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Helmholtz Free Energy for the HGO Model

For the HGO model, the Helmholtz free energy per unit reference volume writes

$$W(\bar{I}_1, \bar{I}_4, J) = W_{iso}(\bar{I}_1, J) + W_{ani}(\bar{I}_1, \bar{I}_4), \tag{1}$$

with an isotropic contribution W_{iso} from generalised Neo-Hookean model (METAFOR)

$$W_{iso}(\bar{I}_1, J) = C_1(\bar{I}_1 - 3) + W(J) = \frac{G}{2}(\bar{I}_1 - 3) + \frac{k_0}{2}\left[(J-1)^2 + \ln^2 J\right], \tag{2}$$

and an anisotropic contribution W_{ani} with n families of fibers which writes [1]

$$W_{ani}(\bar{I}_1, \bar{I}_4) = \frac{k_1}{2k_2}\sum_{\alpha=1}^{n}\left[e^{k_2 < E_\alpha >^2} - 1\right] = \frac{k_1}{2k_2}\sum_{\alpha=1}^{n}\left[e^{k_2 \left\langle d(\bar{I}_1 - 3) + (1-3d)(\bar{I}_4^\alpha - 1)\right\rangle^2} - 1\right], \tag{3}$$

with $\bar{I}_4^\alpha = (\bar{\mathbf{F}}\mathbf{a}^\alpha) \cdot (\bar{\mathbf{F}}\mathbf{a}^\alpha)$ and $\mathbf{a}^\alpha = [a_{\alpha x},\ a_{\alpha y},\ a_{\alpha z}]$

Helmholtz Free Energy for the HGO Model *(ctd.)*

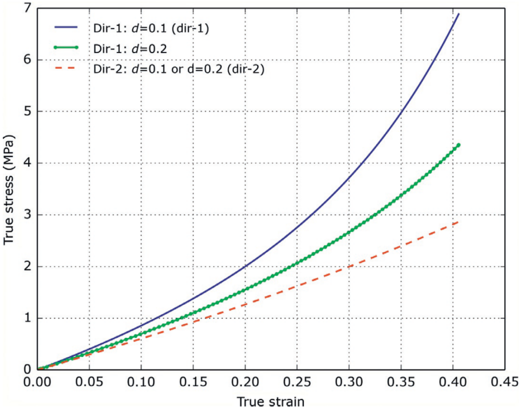
$d \in \left[0;\ \frac{1}{3}\right]$ is a parameter accounting for fiber dispersion, with $d = 0$ corresponding to perfectly aligned fibers whilst $d = \frac{1}{3}$ corresponds to randomly aligned fibers. In the literature, it is more commonly assumed that the fibers are aligned ($d = 0$).

Due to the Macaulay brackets $\langle ... \rangle$, the anisotropic contribution only affects the traction behaviour of the material as $W_{ani} = 0$ if $E_\alpha = 0$.

$$\langle E_\alpha \rangle = \begin{cases} E_\alpha & \text{if } E_\alpha \geq 0 \\ 0 & \text{if } E_\alpha < 0 \end{cases} \tag{4}$$

This means that the fibers do not play any role in the compression behaviour of the material, as $E_\alpha \neq 0$ implies $\bar{I}_4^\alpha > 1$ when $d = 0$.

HGO Model - Stress-Strain Curve



Predicted stress-strain response for the HGO model [1]

$$C_1 = 1\text{MPa}$$

$$k_0 = 20\text{MPa}$$

$$k_1 = 1\text{MPa}$$

$$k_2 = 1$$

$$\mathbf{a}^1 = [1,\ 0,\ 0]$$

Isotropic PK2 Stress Tensor

From the isotropic free energy W_{iso}, the isotropic PK2 stress tensor $\mathbf{S}_{iso}$ writes

$$\mathbf{S}_{iso} = 2J^{-\frac{2}{3}} \left[\frac{\partial W_{iso}}{\partial \bar{I}_1} \mathbf{I} + \frac{\partial W_{iso}}{\partial J} \frac{J}{2} \bar{\mathbf{C}}^{-1} \right]. \tag{5}$$

From Eqn. (2), we have

$$\frac{\partial W_{iso}}{\partial \bar{I}_1} = C_1 \text{ and } \frac{\partial W_{iso}}{\partial J} = -\frac{2}{3} C_1 J^{-\frac{5}{3}} I_1 + 2\frac{k_0}{2}(J-1) + 2\frac{k_0}{2}\frac{\ln J}{J}. \tag{6}$$

Injecting into Eqn. (5) and using $\bar{I}_1 = J^{-\frac{2}{3}} I_1$,

$$\begin{aligned} \mathbf{S}_{iso} &= 2J^{-\frac{2}{3}} \left[C_1 \mathbf{I} + \left(-\frac{2}{3} C_1 J^{-\frac{5}{3}} I_1 + 2\frac{k_0}{2}[J-1] + 2\frac{k_0}{2}\frac{\ln J}{J} \right) \frac{J}{2} \bar{\mathbf{C}}^{-1} \right] \\ &= 2J^{-\frac{2}{3}} \left[C_1 \mathbf{I} + \left(-\frac{2}{3} C_1 \bar{I}_1 + k_0 J[J-1] + k_0 \ln J \right) \frac{1}{2} \bar{\mathbf{C}}^{-1} \right] \end{aligned} \tag{7}$$

Isotropic Cauchy Stress Tensor

The isotropic Cauchy stress tensor $\boldsymbol{\sigma}_{iso}$ is computed from $\mathbf{S}_{iso}$ as

$$\boldsymbol{\sigma}_{iso} = \frac{1}{J}\mathbf{F}\mathbf{S}_{iso}\mathbf{F}^T \tag{8}$$

Injecting Eqn. (7) and using $\bar{\mathbf{B}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^T = J^{-\frac{2}{3}}\mathbf{B}$ and $\bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}} = J^{-\frac{2}{3}}\mathbf{C}$ yields

$$\begin{aligned}
\boldsymbol{\sigma}_{iso} &= \frac{2}{J}J^{-\frac{2}{3}}C_1\mathbf{F}\mathbf{I}\mathbf{F}^T + \left(-\frac{2}{3}C_1\bar{I}_1 + k_0J[J-1] + k_0\ln J\right)\frac{1}{J}J^{-\frac{2}{3}}\mathbf{F}\bar{\mathbf{C}}^{-1}\mathbf{F}^T \\
&= \frac{2}{J}C_1J^{-\frac{2}{3}}\mathbf{B} + \left(-\frac{2}{3}C_1\bar{I}_1 + k_0J[J-1] + k_0\ln J\right)\frac{1}{J}\frac{J^{-\frac{2}{3}}}{J^{-\frac{2}{3}}}\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T
\end{aligned}$$

Isotropic Cauchy Stress Tensor *(ctd.)*

$$
\begin{aligned}
&= 2\frac{C_1}{J}\bar{\mathbf{B}} + \left(-\frac{2}{3}C_1\bar{I}_1 + k_0 J[J-1] + k_0 \ln J\right)\frac{1}{J}\mathbf{I} \\
&= 2\frac{C_1}{J}\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right) + k_0\left([J-1] + \frac{\ln J}{J}\right)\mathbf{I} \\
&= 2\frac{C_1}{J}\left(\bar{\mathbf{B}} - \frac{1}{3}\mathrm{tr}(\bar{\mathbf{B}})\mathbf{I}\right) + k_0\left([J-1] + \frac{\ln J}{J}\right)\mathbf{I} \\
&= 2\frac{C_1}{J}\mathrm{dev}(\bar{\mathbf{B}}) + k_0\left([J-1] + \frac{\ln J}{J}\right)\mathbf{I}
\end{aligned} \tag{9}
$$

From Eqn. (9), we can express the isotropic deviatoric stress $\boldsymbol{s}$ contribution

$$
\boldsymbol{s} = \mathrm{dev}(\boldsymbol{\sigma}_{iso}) = 2\frac{C_1}{J}\mathrm{dev}(\bar{\mathbf{B}}) = \frac{G}{J}\mathrm{dev}(\bar{\mathbf{B}}) \tag{10}
$$

Isotropic Cauchy Stress Tensor *(ctd.)*

and pressure p

$$p = \frac{\mathrm{tr}(\boldsymbol{\sigma}_{iso})}{3} = k_0 \left[(J-1) + \frac{\ln J}{J} \right] \tag{11}$$

Note that in the literature, the pressure term is in most cases

$$p = k_0(J-1), \tag{12}$$

due to the use of $W(J) = \frac{k_0}{2}(J-1)^2$ for the HGO model ($\neq$METAFOR). Therefore, it may be necessary to adapt the value of the bulk modulus k_0 from the literature.

Anisotropic PK2 Stress Tensor

From the anisotropic free energy W_{ani}, the anisotropic PK2 stress tensor $\mathbf{S}_{ani}$ writes

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}} \left[\frac{\partial W_{ani}}{\partial \bar{I}_1} \mathbf{I} + \frac{\partial W_{ani}}{\partial J} \frac{J}{2} \bar{\mathbf{C}}^{-1} + \sum_{\alpha} \frac{\partial W_{ani}}{\partial \bar{I}_4^{\alpha}} \mathbf{M}^{\alpha} \right] \tag{13}$$

From Eqn. (3) and using $\mathbf{M}^{\alpha} = \mathbf{a}^{\alpha} \otimes \mathbf{a}^{\alpha}$, we have

$$\begin{aligned} \frac{\partial W_{ani}}{\partial J} &= \frac{k_1}{2k_2} \sum_{\alpha} 2k_2 \frac{\partial E_{\alpha}}{\partial J} < E_{\alpha} > e^{k_2 < E_{\alpha} >^2} \\ &= k_1 \sum_{\alpha} \left[-\frac{2}{3} d\, J^{-\frac{5}{3}} I_1 - \frac{2}{3} (1 - 3d) J^{-\frac{5}{3}} I_4^{\alpha} \right] < E_{\alpha} > e^{k_2 < E_{\alpha} >^2} \end{aligned} \tag{14}$$

Anisotropic PK2 Stress Tensor *(ctd.)*

$$\frac{\partial W_{ani}}{\partial J}\frac{J}{2} = \frac{k_1}{2}\sum_\alpha \left[-\frac{2}{3}d\,J^{-\frac{2}{3}}I_1 - \frac{2}{3}(1-3d)J^{-\frac{2}{3}}I_4^\alpha\right] < E_\alpha > e^{k_2 < E_\alpha >^2}$$

$$= -\frac{1}{3}k_1\sum_\alpha \left[d\,\bar{I}_1 + (1-3d)\bar{I}_4^\alpha\right] < E_\alpha > e^{k_2 < E_\alpha >^2} \quad (15)$$

$$\frac{\partial W_{ani}}{\partial \bar{I}_1} = \frac{k_1}{2k_2}\sum_\alpha 2k_2\frac{\partial E_\alpha}{\partial \bar{I}_1} < E_\alpha > e^{k_2 < E_\alpha >^2} = k_1\sum_\alpha d < E_\alpha > e^{k_2 < E_\alpha >^2} \quad (16)$$

$$\sum_\alpha \frac{\partial W_{ani}}{\partial \bar{I}_4^\alpha} = \frac{k_1}{2k_2}\sum_\alpha 2k_2\frac{\partial E_\alpha}{\partial \bar{I}_4^\alpha} < E_\alpha > e^{k_2 < E_\alpha >^2} = k_1\sum_\alpha (1-3d) < E_\alpha > e^{k_2 < E_\alpha >^2} \quad (17)$$

Anisotropic PK2 Stress Tensor *(ctd.)*

Injecting into Eqn. (13) yields

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}} k_1 \sum_{\alpha} \left[d\mathbf{I} - \frac{1}{3} \left(d\,\bar{I}_1 + (1-3d)\bar{I}_4^{\alpha} \right) \bar{\mathbf{C}}^{-1} + (1-3d)\mathbf{M}^{\alpha} \right] < E_{\alpha} > e^{k_2 < E_{\alpha} >^2} \quad (18)$$

It is often assumed that all fibers are aligned, resulting in no fiber dispersion factor $d = 0$. Eqn. (18) is then reduced to

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}} k_1 \sum_{\alpha} \left[-\frac{1}{3} \bar{I}_4^{\alpha} \bar{\mathbf{C}}^{-1} + \mathbf{M}^{\alpha} \right] < \bar{I}_4^{\alpha} - 1 > e^{k_2 < \bar{I}_4^{\alpha} - 1 >^2}, \quad (19)$$

which is the most commonly encoutered form of the HGO model in the literature.

Anisotropic Cauchy Stress Tensor

The anisotropic Cauchy stress tensor $\boldsymbol{\sigma}_{ani}$ is computed from $\mathbf{S}_{iso}$ as

$$\boldsymbol{\sigma}_{ani} = \frac{1}{J}\mathbf{F}\mathbf{S}_{ani}\mathbf{F}^T. \tag{20}$$

Injecting Eqn. (18) and using $\bar{\mathbf{N}}^\alpha = \bar{\mathbf{F}}\mathbf{M}^\alpha\bar{\mathbf{F}}^T = J^{-\frac{2}{3}}\mathbf{N}^\alpha$ yields

$$\begin{aligned}\boldsymbol{\sigma}_{ani} &= 2J^{-\frac{2}{3}}\frac{k_1}{J}\sum_i\left[d\mathbf{B} - \frac{1}{3}\left(d\,\bar{I}_1 + (1-3d)\bar{I}_4^\alpha\right)J^{\frac{2}{3}}\mathbf{I} + (1-3d)\mathbf{N}^\alpha\right] < E_\alpha > e^{k_2 < E_\alpha >^2} \\ &= 2\frac{k_1}{J}\sum_\alpha\left[d\bar{\mathbf{B}} - \frac{1}{3}\left(d\,\bar{I}_1 + (1-3d)\bar{I}_4^\alpha\right)\mathbf{I} + (1-3d)\bar{\mathbf{N}}^\alpha\right] < E_\alpha > e^{k_2 < E_\alpha >^2}\end{aligned}$$

Anisotropic Cauchy Stress Tensor *(ctd.)*

$$= 2\frac{k_1}{J}\sum_\alpha \left[d\left(\bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1\mathbf{I}\right) + (1-3d)\left(\bar{\mathbf{N}}^\alpha - \frac{1}{3}\bar{I}_4^\alpha\mathbf{I}\right)\right] < E_\alpha > e^{k_2 < E_\alpha >^2}$$

$$= 2\frac{k_1}{J}\sum_\alpha \left[d\,\mathrm{dev}(\bar{\mathbf{B}}) + (1-3d)\mathrm{dev}(\bar{\mathbf{N}}^\alpha)\right] < E_\alpha > e^{k_2 < E_\alpha >^2} \tag{21}$$

Eqn. (21) can also be particularised in the case fibers are perfectly aligned ($d = 0$)

$$\boldsymbol{\sigma}_{ani} = 2\frac{k_1}{J}\sum_\alpha \mathrm{dev}(\bar{\mathbf{N}}^\alpha) < \bar{I}_4^\alpha - 1 > e^{k_2 < \bar{I}_4^\alpha - 1 >^2} \tag{22}$$

The anisotropic Cauchy stress tensor only contributes to the deviatoric stress $\boldsymbol{s}$.

Anisotropic Elasticity Tensor for Cauchy Stresses

The anisotropic part of elasticity tensor $\mathbb{H}^{ani}$ from Gasser and Holzapfel [2] is expressed for Kirchhoff stresses $\boldsymbol{\kappa}$

$$\mathbb{H}^{ani,\boldsymbol{\kappa}} = 2\sum_{\alpha} \frac{\partial \boldsymbol{\kappa}_{ani}}{\partial \mathbf{g}} = 4\sum_{\alpha}\Bigg[\frac{k_1}{3}\left\langle \bar{I}_4^{\alpha} - 1\right\rangle e^{k_2<\bar{I}_4^{\alpha}-1>^2}\left(\bar{I}_4^{\alpha}[\mathbb{I} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}] - \bar{\mathbf{N}}^{\alpha}\otimes\mathbf{I} - \mathbf{I}\otimes\bar{\mathbf{N}}^{\alpha}\right)$$

$$+k_1\left(1 + 2k_2\left\langle \bar{I}_4^{\alpha} - 1\right\rangle^2\right) e^{k_2<\bar{I}_4^{\alpha}-1>^2}\operatorname{dev}(\bar{\mathbf{N}}^{\alpha})\otimes\operatorname{dev}(\bar{\mathbf{N}}^{\alpha})\Bigg], \qquad (23)$$

The elasticity tensor for Cauchy stresses is obtained as

$$\mathbb{H}^{ani,\boldsymbol{\sigma}} = \frac{1}{J}\mathbb{H}^{ani,\boldsymbol{\kappa}}. \qquad (24)$$

Anisotropic Elasticity Tensor for Cauchy Stresses *(ctd.)*

⚠ Eqn. (23) is only valid for $d = 0$!

Using Voigt notation, parts of Eqn. (23) now write

$$\mathbb{I} + \frac{1}{3}\mathbf{I} \otimes \mathbf{I} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{1}{3}\delta_{ij}\delta_{kl} \tag{25}$$

$$\bar{\mathbf{N}}^{\alpha} \otimes \mathbf{I} = \bar{\mathrm{N}}^{\alpha}_{ij}\delta_{kl} \tag{26}$$

$$\mathbf{I} \otimes \bar{\mathbf{N}}^{\alpha} = \delta_{ij}\bar{\mathrm{N}}^{\alpha}_{kl} \tag{27}$$

$$\begin{aligned} \mathrm{dev}(\bar{\mathbf{N}}^{\alpha}) \otimes \mathrm{dev}(\bar{\mathbf{N}}^{\alpha}) &= \left(\bar{\mathrm{N}}^{\alpha}_{ij} - \frac{1}{3}\bar{\mathrm{N}}^{\alpha}_{qq}\delta_{ij}\right)\left(\bar{\mathrm{N}}^{\alpha}_{kl} - \frac{1}{3}\bar{\mathrm{N}}^{\alpha}_{qq}\delta_{kl}\right) \\ &= \bar{\mathrm{N}}^{\alpha}_{ij}\bar{\mathrm{N}}^{\alpha}_{kl} - \frac{1}{3}\bar{\mathrm{N}}^{\alpha}_{qq}\bar{\mathrm{N}}^{\alpha}_{ij}\delta_{kl} - \frac{1}{3}\bar{\mathrm{N}}^{\alpha}_{qq}\delta_{ij}\bar{\mathrm{N}}^{\alpha}_{kl} + \frac{1}{9}(\bar{\mathrm{N}}^{\alpha}_{qq})^2\delta_{ij}\delta_{kl} \end{aligned} \tag{28}$$

Anisotropic Elasticity Tensor for Cauchy Stresses *(ctd.)*

To further simplify the notation, stress functions ψ'_α and ψ''_α are also introduced

$$\psi'_\alpha = k_1 \left\langle \bar{I}_4^\alpha - 1 \right\rangle e^{k_2 < \bar{I}_4^\alpha - 1 >^2} \tag{29}$$

$$\psi''_\alpha = k_1 \left(1 + 2k_2 \left\langle \bar{I}_4^\alpha - 1 \right\rangle^2 \right) e^{k_2 < \bar{I}_4^\alpha - 1 >^2} \tag{30}$$

Injecting Eqn. (24), the elasticity tensor for Cauchy stresses writes in Voigt notation

$$\begin{aligned} \mathbb{H}^{ani,\boldsymbol{\sigma}}_{ijkl} = & \frac{4}{J} \sum_\alpha \Bigg[\frac{1}{3} \psi'_\alpha \left(\bar{I}_4^\alpha [\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{1}{3}\delta_{ij}\delta_{kl}] - \bar{\mathrm{N}}^\alpha_{ij}\delta_{kl} - \delta_{ij}\bar{\mathrm{N}}^\alpha_{kl} \right) \\ & + \psi''_\alpha \left(\bar{\mathrm{N}}^\alpha_{ij}\bar{\mathrm{N}}^\alpha_{kl} - \frac{1}{3}\bar{\mathrm{N}}^\alpha_{qq}\bar{\mathrm{N}}^\alpha_{ij}\delta_{kl} - \frac{1}{3}\bar{\mathrm{N}}^\alpha_{qq}\delta_{ij}\bar{\mathrm{N}}^\alpha_{kl} + \frac{1}{9}(\bar{\mathrm{N}}^\alpha_{qq})^2\delta_{ij}\delta_{kl} \right) \Bigg] \end{aligned} \tag{31}$$

Anisotropic Elasticity Tensor for Cauchy Stresses *(ctd.)*

Arranging the terms accordingly

$$\mathbb{H}_{ijkl}^{ani,\boldsymbol{\sigma}} = \frac{4}{J}\sum_{\alpha}\Bigg[\frac{1}{6}\psi'_\alpha \bar{I}_4^\alpha\,\delta_{ik}\delta_{jl} + \frac{1}{6}\psi'_\alpha \bar{I}_4^\alpha\,\delta_{il}\delta_{jk} + \frac{1}{9}\left(\psi'_\alpha \bar{I}_4^\alpha + \psi''_\alpha (\bar{\mathsf{N}}_{qq}^\alpha)^2\right)\delta_{ij}\delta_{kl}$$
$$+\psi''_\alpha \bar{\mathsf{N}}_{ij}^\alpha \bar{\mathsf{N}}_{kl}^\alpha - \frac{1}{3}\left(\psi'_\alpha + \psi''_\alpha \bar{\mathsf{N}}_{qq}^\alpha\right)\bar{\mathsf{N}}_{ij}^\alpha \delta_{kl} - \frac{1}{3}\left(\psi'_\alpha + \psi''_\alpha \bar{\mathsf{N}}_{qq}^\alpha\right)\delta_{ij}\bar{\mathsf{N}}_{kl}^\alpha\Bigg] \tag{32}$$

and further simplifying the notation

$$\mathbb{H}_{ijkl}^{ani,\boldsymbol{\sigma}} = \frac{4}{J}\sum_{\alpha}\left[A_1\,\delta_{ik}\delta_{jl} + A_1\,\delta_{il}\delta_{jk} + A_2\,\delta_{ij}\delta_{kl} + \psi''_\alpha \bar{\mathsf{N}}_{ij}^\alpha \bar{\mathsf{N}}_{kl}^\alpha + A_3\,\bar{\mathsf{N}}_{ij}^\alpha \delta_{kl} + A_3 \delta_{ij}\bar{\mathsf{N}}_{kl}^\alpha\right] \tag{33}$$

with $A_1 = \frac{1}{6}\psi'_\alpha \bar{I}_4^\alpha$ | $A_2 = \frac{1}{9}\left(\psi'_\alpha \bar{I}_4^\alpha + \psi''_\alpha (\bar{\mathsf{N}}_{qq}^\alpha)^2\right)$ | $A_3 = -\frac{1}{3}\left(\psi'_\alpha + \psi''_\alpha \bar{\mathsf{N}}_{qq}^\alpha\right)$

Pk2 Anisotropic Elasticity Tensor for $d = 0$

The anisotropic contribution to the elasticity tensor for Pk2 stress $\mathbb{H}^{\mathbf{S}}$ writes

$$\mathbb{H}^{\mathbf{S}}_{ani} = 2\frac{\partial \mathbf{S}_{ani}}{\partial \mathbf{C}} = 4\frac{\partial^2 W_{ani}}{\partial \mathbf{C}\partial \mathbf{C}} \tag{34}$$

If $d = 0$, the anisotropic contribution from Pk2 stress writes

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}}k_1 \sum_{\alpha} \left[-\frac{1}{3}\bar{I}_4^{\alpha}\bar{\mathbf{C}}^{-1} + \mathbf{M}^{\alpha} \right] < \bar{I}_4^{\alpha} - 1 > e^{k_2 <\bar{I}_4^{\alpha} - 1>^2} \stackrel{\text{(Eq. 29)}}{=} 2J^{-\frac{2}{3}} \sum_{\alpha} \operatorname{dev}\left(\mathbf{M}^{\alpha}\right) \psi'_{\alpha} \tag{35}$$

Therefore,

$$\mathbb{H}^{\mathbf{S}}_{ani} = 4\sum_{\alpha} \left[\operatorname{dev}\left(\mathbf{M}^{\alpha}\right) \psi'_{\alpha} \otimes \frac{\partial}{\partial \mathbf{C}} \left(J^{-\frac{2}{3}}\right) + J^{-\frac{2}{3}} \psi'_{\alpha} \frac{\partial}{\partial \mathbf{C}} \left(\operatorname{dev}\left(\mathbf{M}^{\alpha}\right)\right) + J^{-\frac{2}{3}} \operatorname{dev}\left(\mathbf{M}^{\alpha}\right) \otimes \frac{\partial \psi'_{\alpha}}{\partial \mathbf{C}} \right] \tag{36}$$

Pk2 Anisotropic Elasticity Tensor for $d = 0$ *(ctd.)*

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{C}} \left(\operatorname{dev}\left(\mathbf{M}^{\alpha}\right)\right) &= \frac{\partial \mathbf{M}^{\alpha}}{\partial \mathbf{C}} - \frac{1}{3} \frac{\partial}{\partial \mathbf{C}} \left(\bar{I}_4^{\alpha} \bar{\mathbf{C}}^{-1}\right) && (37) \\
&= 0 - \frac{1}{3} \left[\bar{\mathbf{C}}^{-1} \otimes \frac{\partial \bar{I}_4^{\alpha}}{\partial J} \frac{J}{2} \mathbf{C}^{-1} + \bar{\mathbf{C}}^{-1} \otimes \frac{\partial \bar{I}_4^{\alpha}}{\partial I_4^{\alpha}} \mathbf{M}^{\alpha} + \bar{I}_4^{\alpha} \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}}\right] \\
&= -\frac{1}{3} \left[\bar{\mathbf{C}}^{-1} \otimes \left(-\frac{1}{3} J^{-\frac{2}{3}} \bar{I}_4^{\alpha}\right) \mathbf{C}^{-1} + \bar{\mathbf{C}}^{-1} \otimes J^{-\frac{2}{3}} \mathbf{M}^{\alpha} + \bar{I}_4^{\alpha} \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}} : \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}\right] \\
&= -\frac{1}{3} \left[J^{-\frac{2}{3}} \bar{\mathbf{C}}^{-1} \otimes \left(\mathbf{M}^{\alpha} - \frac{1}{3} \bar{I}_4^{\alpha} \bar{\mathbf{C}}^{-1}\right) + \bar{I}_4^{\alpha} \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}} : \left(\mathbb{I}^{sym} - \frac{1}{3} \mathbf{C} \otimes \mathbf{C}^{-1}\right)\right] \\
&= -\frac{1}{3} \left[\mathbf{C}^{-1} \otimes \operatorname{dev}\left(\mathbf{M}^{\alpha}\right) - J^{\frac{2}{3}} \bar{I}_4^{\alpha} \left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right)\right] && (38)
\end{aligned}
$$

Pk2 Anisotropic Elasticity Tensor for $d = 0$ *(ctd.)*

$$\frac{\partial}{\partial \mathbf{C}}\left(J^{-\frac{2}{3}}\right) = \frac{\partial}{\partial J}\left(J^{-\frac{2}{3}}\right)\frac{\partial J}{\partial \mathbf{C}} \tag{39}$$

$$= -\frac{2}{3}J^{-\frac{5}{3}}\frac{J}{2}\mathbf{C}^{-1} = -\frac{1}{3}J^{-\frac{2}{3}}\mathbf{C}^{-1} \tag{40}$$

$$\frac{\partial \psi'_\alpha}{\partial \mathbf{C}} = \frac{\partial \psi'_\alpha}{\partial I_1}\mathbf{I} + \frac{\partial \psi'_\alpha}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial \psi'_\alpha}{\partial I_4^\alpha}\mathbf{M}^\alpha \tag{41}$$

$$= \frac{\partial \psi'_\alpha}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial \psi'_\alpha}{\partial I_4^\alpha}\mathbf{M}^\alpha \tag{42}$$

Pk2 Anisotropic Elasticity Tensor for $d = 0$ *(ctd.)*

$$\frac{\partial \psi'_\alpha}{\partial J}\frac{J}{2} = \frac{J}{2}\frac{\partial}{\partial J}\left(k_1 \left\langle \bar{I}_4^\alpha - 1\right\rangle e^{k_2\left\langle \bar{I}_4^\alpha - 1\right\rangle^2}\right) \tag{43}$$

$$= \frac{J}{2}\left[-\frac{2}{3}J^{-1}\bar{I}_4^\alpha e^{k_2\left\langle \bar{I}_4^\alpha - 1\right\rangle^2} - \frac{2}{3}J^{-1}\bar{I}_4^\alpha 2k_2 \left\langle \bar{I}_4^\alpha - 1\right\rangle^2 e^{k_2\left\langle \bar{I}_4^\alpha - 1\right\rangle^2}\right]$$

$$\overset{\text{(Eq. 30)}}{=} -\frac{1}{3}\bar{I}_4^\alpha \psi''_\alpha \tag{44}$$

$$\frac{\partial \psi'_\alpha}{\partial I_4^\alpha} = \frac{\partial \psi'_\alpha}{\partial I_4^\alpha}\left(k_1 \left\langle \bar{I}_4^\alpha - 1\right\rangle e^{k_2\left\langle \bar{I}_4^\alpha - 1\right\rangle^2}\right) \tag{45}$$

$$= k_1 J^{-\frac{2}{3}} e^{k_2\left\langle \bar{I}_4^\alpha - 1\right\rangle^2} + k_1 J^{-\frac{2}{3}} k_1 \left\langle \bar{I}_4^\alpha - 1\right\rangle^2 e^{k_2\left\langle \bar{I}_4^\alpha - 1\right\rangle^2} \overset{\text{(Eq. 30)}}{=} J^{-\frac{2}{3}}\psi''_\alpha \tag{46}$$

Pk2 Anisotropic Elasticity Tensor for $d = 0$ *(ctd.)*

$$\begin{aligned}
\mathbb{H}^{\mathbf{S}}_{ani} &= 4\sum_{\alpha}\Big\{\psi'_\alpha \operatorname{dev}(\mathbf{M}^\alpha)\otimes\left(-\frac{1}{3}J^{-\frac{2}{3}}\right)\mathbf{C}^{-1} - \frac{1}{3}J^{-\frac{2}{3}}\psi'_\alpha\Big[\mathbf{C}^{-1}\otimes\operatorname{dev}(\mathbf{M}^\alpha) \\
&\quad -J^{\frac{2}{3}}\bar{I}^\alpha_4\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1}-\frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\right)\Big] + J^{-\frac{2}{3}}\operatorname{dev}(\mathbf{M}^\alpha)\otimes\left[\psi''_\alpha J^{-\frac{2}{3}}\mathbf{M}^\alpha - \frac{1}{3}\psi''_\alpha\bar{I}^\alpha_4\mathbf{C}^{-1}\right]\Big\} \\
&= 4\sum_{\alpha}\Big\{-\frac{1}{3}\psi'_\alpha J^{-\frac{2}{3}}\mathbf{M}^\alpha\otimes\mathbf{C}^{-1} - \frac{1}{3}\psi'_\alpha J^{-\frac{2}{3}}\left(-\frac{1}{3}\bar{I}^\alpha_4\right)\bar{\mathbf{C}}^{-1}\otimes\mathbf{C}^{-1} - \frac{1}{3}\psi'_\alpha J^{-\frac{2}{3}}\mathbf{C}^{-1}\otimes\mathbf{M}^\alpha \\
&\quad -\frac{1}{3}\psi'_\alpha J^{-\frac{2}{3}}\left(-\frac{1}{3}\bar{I}^\alpha_4\right)\mathbf{C}^{-1}\otimes\bar{\mathbf{C}}^{-1} + \frac{1}{3}\psi'_\alpha\bar{I}^\alpha_4\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1}-\frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\right) \\
&\quad +\psi''_\alpha J^{-\frac{4}{3}}\operatorname{dev}(\mathbf{M}^\alpha)\otimes\left[\mathbf{M}^\alpha-\frac{1}{3}\bar{I}^\alpha_4\bar{\mathbf{C}}^{-1}\right]\Big\}
\end{aligned} \tag{47}$$

Pk2 Anisotropic Elasticity Tensor for $d = 0$ *(ctd.)*

$$
\begin{aligned}
\mathbb{H}^{\mathbf{S}}_{ani} = 4 \sum_{\alpha} \Bigg\{ & -\frac{1}{3}\psi'_\alpha J^{-\frac{2}{3}} \mathbf{M}^\alpha \otimes \mathbf{C}^{-1} - \frac{1}{3}\psi'_\alpha \left(-\frac{1}{3}\bar{I}^\alpha_4\right) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - \frac{1}{3}\psi'_\alpha J^{-\frac{2}{3}} \mathbf{C}^{-1} \otimes \mathbf{M}^\alpha \\
& - \frac{1}{3}\psi'_\alpha \left(-\frac{1}{3}\bar{I}^\alpha_4\right) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + \frac{1}{3}\psi'_\alpha \bar{I}^\alpha_4 \left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right) \\
& + \psi''_\alpha J^{-\frac{4}{3}} \operatorname{dev}(\mathbf{M}^\alpha) \otimes \operatorname{dev}(\mathbf{M}^\alpha) \Bigg\} \\
= 4 \sum_{\alpha} \Bigg\{ & \frac{1}{3}\psi'_\alpha \left[\bar{I}^\alpha_4 \left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} + \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right) - J^{-\frac{2}{3}} \mathbf{M}^\alpha \otimes \mathbf{C}^{-1} - J^{-\frac{2}{3}} \mathbf{C}^{-1} \otimes \mathbf{M}^\alpha\right] \\
& + \psi''_\alpha J^{-\frac{4}{3}} \operatorname{dev}(\mathbf{M}^\alpha) \otimes \operatorname{dev}(\mathbf{M}^\alpha) \Bigg\} \qquad (48)
\end{aligned}
$$

Cauchy Anisotropic Elasticity Tensor for $d = 0$

The anisotropic contribution to the elasticity tensor for Cauchy stress $\mathbb{H}^{\sigma}_{ani}$ is obtained with a push-forward operation of $\mathbb{H}^{\mathbf{S}}_{ani}$ to the current configuration

$$\mathbb{H}^{\sigma}_{ani} = \frac{1}{J}\mathbf{F}(\mathbf{F}\mathbb{H}^{\mathbf{S}_{ani}}\mathbf{F}^{T})\mathbf{F}^{T} \tag{49}$$

Using Eqn 48, $\mathbb{H}^{\sigma}_{ani}$ finally writes

$$\mathbb{H}^{\sigma}_{ani} = \frac{4}{J}\sum_{\alpha}\left\{\frac{1}{3}\psi'_{\alpha}\left[\bar{I}^{\alpha}_{4}\left(\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right) - \bar{\mathbf{N}}^{\alpha}\otimes\mathbf{I} - \mathbf{I}\otimes\bar{\mathbf{N}}^{\alpha}\right] + \psi''_{\alpha}\mathrm{dev}\left(\bar{\mathbf{N}}^{\alpha}\right)\otimes\mathrm{dev}\left(\bar{\mathbf{N}}^{\alpha}\right)\right\}, \tag{50}$$

which presents a minor and major symmetry and is the same expression found by Gasser and Holzapfel [2] in Eqn. 23.

Pk2 Anisotropic Elasticity Tensor for $d \neq 0$

If $d \neq 0$, the anisotropic contribution from Pk2 stress writes

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}} k_1 \sum_{\alpha} \left[d\mathbf{I} - \frac{1}{3} \left(d\,\bar{I}_1 + (1-3d)\bar{I}_4^{\alpha} \right) \bar{\mathbf{C}}^{-1} + (1-3d)\mathbf{M}^{\alpha} \right] < E_{\alpha} > e^{k_2 < E_{\alpha} >^2} \quad (51)$$

$$= 2J^{-\frac{2}{3}} \sum_{\alpha} \left\{ d \left(\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1} \right) + (1-3d)\, \mathrm{dev}\left(\mathbf{M}^{\alpha}\right) \right\} < E_{\alpha} > e^{k_2 < E_{\alpha} >^2} \quad (52)$$

Let us introduce the following stress function

$$\psi'_{\alpha} = k_1 < E_{\alpha} > e^{k_2 < E_{\alpha} >^2}, \quad (53)$$

such that

$$\mathbf{S}_{ani} = 2J^{-\frac{2}{3}} \sum_{\alpha} \psi'_{\alpha} \left\{ d \left(\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1} \right) + (1-3d)\, \mathrm{dev}\left(\mathbf{M}^{\alpha}\right) \right\} \quad (54)$$

Pk2 Anisotropic Elasticity Tensor for $d \neq 0$ *(ctd.)*

Therefore,

$$\mathbb{H}^{\mathbf{S}}_{ani} = 4(1-3d)\sum_{\alpha}\Bigg[\operatorname{dev}(\mathbf{M}^{\alpha})\,\psi'_{\alpha}\otimes\frac{\partial}{\partial\mathbf{C}}\left(J^{-\frac{2}{3}}\right) + J^{-\frac{2}{3}}\psi'_{\alpha}\frac{\partial}{\partial\mathbf{C}}\left(\operatorname{dev}(\mathbf{M}^{\alpha})\right)$$
$$+J^{-\frac{2}{3}}\operatorname{dev}(\mathbf{M}^{\alpha})\otimes\frac{\partial}{\partial\mathbf{C}}\left(\psi'_{\alpha}\right)\Bigg] + 4d\sum_{\alpha}\Bigg[\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\psi'_{\alpha}\otimes\frac{\partial}{\partial\mathbf{C}}\left(J^{-\frac{2}{3}}\right)$$
$$+J^{-\frac{2}{3}}\psi'_{\alpha}\frac{\partial}{\partial\mathbf{C}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right) + J^{-\frac{2}{3}}\left(\mathbf{I}-\frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1}\right)\otimes\frac{\partial\psi'_{\alpha}}{\partial\mathbf{C}}\Bigg] \qquad (55)$$

Pk2 Anisotropic Elasticity Tensor for $d \neq 0$ *(ctd.)*

$$\frac{\partial}{\partial \mathbf{C}}\left(\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1}\right) = \frac{\partial \mathbf{I}}{\partial \mathbf{C}} - \frac{1}{3}\frac{\partial}{\partial \mathbf{C}}\left(\bar{I}_1 \bar{\mathbf{C}}^{-1}\right) \tag{56}$$

$$\begin{aligned} &= 0 - \frac{1}{3}\left[\bar{\mathbf{C}}^{-1} \otimes \frac{\partial \bar{I}_1}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \bar{\mathbf{C}}^{-1} \otimes \frac{\partial \bar{I}_1}{\partial I_1}\mathbf{I} + \bar{I}_1 \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}}\right] \\ &= -\frac{1}{3}\left[\bar{\mathbf{C}}^{-1} \otimes \left(-\frac{1}{3}J^{-\frac{2}{3}}\bar{I}_1\right)\mathbf{C}^{-1} + \bar{\mathbf{C}}^{-1} \otimes J^{-\frac{2}{3}}\mathbf{I} + \bar{I}_1 \frac{\partial \bar{\mathbf{C}}^{-1}}{\partial \bar{\mathbf{C}}} : \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}\right] \\ &= -\frac{1}{3}\left[\mathbf{C}^{-1} \otimes \left(\mathbf{I} - \frac{1}{3}\bar{I}_1 \bar{\mathbf{C}}^{-1}\right) - J^{\frac{2}{3}}\bar{I}_1\left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right)\right] \end{aligned} \tag{57}$$

Pk2 Anisotropic Elasticity Tensor for $d \neq 0$ *(ctd.)*

$$\frac{\partial \psi'_\alpha}{\partial \mathbf{C}} = \frac{\partial \psi'_\alpha}{\partial I_1}\mathbf{I} + \frac{\partial \psi'_\alpha}{\partial J}\frac{J}{2}\mathbf{C}^{-1} + \frac{\partial \psi'_\alpha}{\partial I_4^\alpha}\mathbf{M}^\alpha \tag{58}$$

To simplify the notation, another stress function is introduced

$$\psi''_\alpha = k_1\left(1 + 2k_2 < E_\alpha >^2\right)\, e^{k_2 < E_\alpha >^2}, \tag{59}$$

Eqn 58 then writes

$$\frac{\partial \psi'_\alpha}{\partial \mathbf{C}} = dJ^{-\frac{2}{3}}\psi''_\alpha \mathbf{I} - \frac{1}{3}\psi''_\alpha\left[d\bar{\mathbf{I}}_1 + (1-3d)\bar{\mathbf{I}}_4^\alpha\right]\mathbf{C}^{-1} + (1-3d)J^{-\frac{2}{3}}\psi''_\alpha \mathbf{M}^\alpha \tag{60}$$

Pk2 Anisotropic Elasticity Tensor for $d \neq 0$ (ctd.)

After further arranging the terms and using $E'_\alpha = d\bar{I}_1 + (1-3d)\bar{I}_4^\alpha$, $\mathbb{H}^{\mathbf{S}}_{ani}$ finally writes

$$\begin{aligned}
\mathbb{H}^{\mathbf{S}}_{ani} = 4\sum_\alpha \Bigg\{ \frac{1}{3}\psi'_\alpha \Bigg[& E'_\alpha \left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} + \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \right) - dJ^{-\frac{2}{3}}\mathbf{I} \otimes \mathbf{C}^{-1} - dJ^{-\frac{2}{3}}\mathbf{C}^{-1} \otimes \mathbf{I} \\
& -(1-3d)J^{-\frac{2}{3}}\mathbf{M}^\alpha \otimes \mathbf{C}^{-1} - (1-3d)J^{-\frac{2}{3}}\mathbf{C}^{-1} \otimes \mathbf{M}^\alpha \Bigg] + (1-3d)^2\psi''_\alpha J^{-\frac{4}{3}} \operatorname{dev}(\mathbf{M}^\alpha) \otimes \operatorname{dev}(\mathbf{M}^\alpha) \\
& + d^2\psi''_\alpha J^{-\frac{4}{3}} \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1} \right) \otimes \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1} \right) + (1-3d)d\psi''_\alpha J^{-\frac{4}{3}} \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1} \right) \otimes \operatorname{dev}(\mathbf{M}^\alpha) \\
& +(1-3d)d\psi''_\alpha J^{-\frac{4}{3}} \operatorname{dev}(\mathbf{M}^\alpha) \otimes \left(\mathbf{I} - \frac{1}{3}\bar{I}_1\bar{\mathbf{C}}^{-1} \right) \Bigg\}
\end{aligned} \tag{61}$$

Note that for $d = 0$, Eqn 48 is recovered.

Cauchy Anisotropic Elasticity Tensor for $d \neq 0$

The anisotropic contribution to the elasticity tensor for Cauchy stress $\mathbb{H}^{\sigma}_{ani}$ is obtained with a push-forward operation of $\mathbb{H}^{\mathbf{S}}_{ani}$ to the current configuration

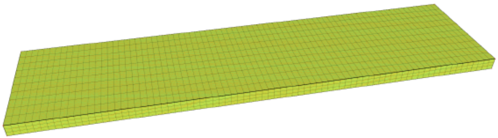
$$\mathbb{H}^{\sigma}_{ani} = \frac{1}{J}\mathbf{F}(\mathbf{F}\mathbb{H}^{\mathbf{S}_{ani}}\mathbf{F}^{T})\mathbf{F}^{T} \tag{62}$$

Using Eqn 61, $\mathbb{H}^{\sigma}_{ani}$ finally writes

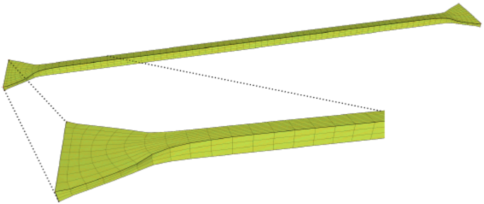
$$\begin{aligned}\mathbb{H}^{\sigma}_{ani} = \frac{4}{J}\sum_{\alpha}\Big\{ & \psi'_{\alpha}\left[E'_{\alpha}\left(\mathbb{I}^{sym} + \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right) - d\left(\mathbf{I}\otimes\bar{\mathbf{B}} + \bar{\mathbf{B}}\otimes\mathbf{I}\right) - (1-3d)\left(\mathbf{I}\otimes\bar{\mathbf{N}}^{\alpha} + \bar{\mathbf{N}}^{\alpha}\otimes\mathbf{I}\right)\right] \\ & + (1-3d)^2\psi''_{\alpha}\,\mathrm{dev}\left(\bar{\mathbf{N}}^{\alpha}\right)\otimes\mathrm{dev}\left(\bar{\mathbf{N}}^{\alpha}\right) + d^2\psi''_{\alpha}\,\mathrm{dev}\left(\bar{\mathbf{B}}\right)\otimes\mathrm{dev}\left(\bar{\mathbf{B}}\right) \\ & + d(1-3d)\psi''_{\alpha}\,\mathrm{dev}\left(\bar{\mathbf{B}}\right)\otimes\mathrm{dev}\left(\bar{\mathbf{N}}^{\alpha}\right) + d(1-3d)\psi''_{\alpha}\,\mathrm{dev}\left(\bar{\mathbf{N}}^{\alpha}\right)\otimes\mathrm{dev}\left(\bar{\mathbf{B}}\right)\Big\}\end{aligned} \tag{63}$$

Metafor Implementation and Tests

Tension of an arterial tissue (2 fiber families at 70° cross-orientation) in 3D from Sarah Hault's Master Thesis [3] (original test from Peyraut *et. al.* [4])



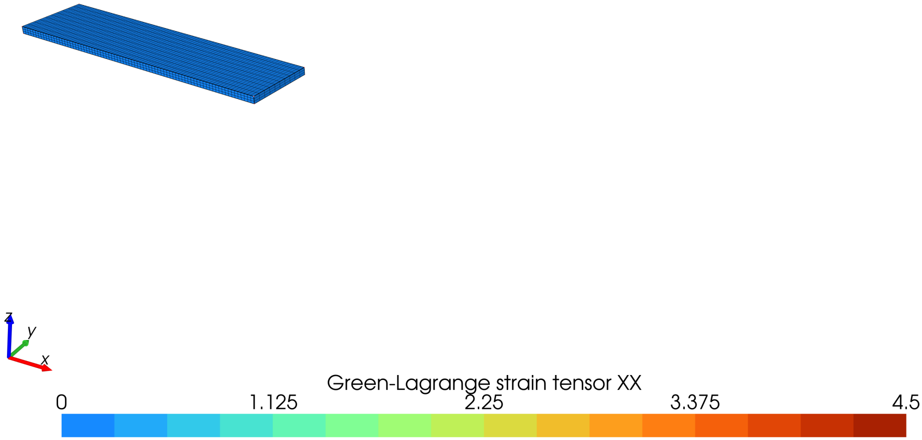
Initial configuration



Final configuration (300% stretch)

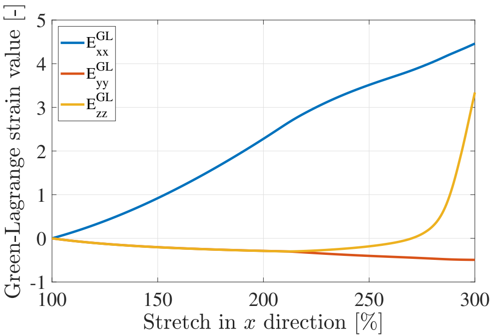
Results and convergence are similar whether a numerical, semi-numerical or analytical stiffness matrix is used in Metafor. This has also been tested using a 2D/3D mono-elements.

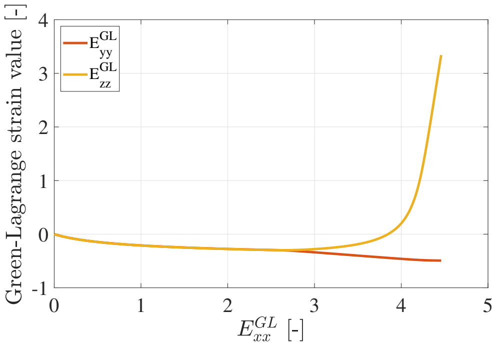
Metafor Implementation and Tests *(ctd.)*



Metafor Implementation and Tests *(ctd.)*

Measure of the mean Green-Lagrange strain value at the center of the piece:





Green-Lagrange strain measures from Metafor (Cauchy HGO material)

Metafor Implementation and Tests *(ctd.)*

(Useful reminders)

Interpretation of the Green-Lagrange strain value:

$$\begin{aligned} E^{GL} > 0 \quad &\rightarrow \text{Elongation} \\ E^{GL} < 0 \quad &\rightarrow \text{Contraction} \end{aligned} \tag{64}$$

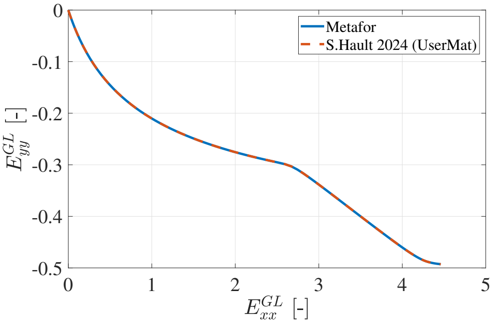
In simple tension, the Green-Lagrange strain writes

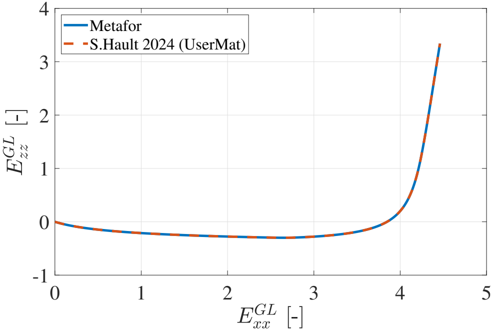
$$E^{GL} = \frac{1}{2}\frac{l^2 - l_0^2}{l_0^2}, \tag{65}$$

which is a non-linear expression ⚠ .

Metafor Implementation and Tests (ctd.)

Comparison of the mean Green-Lagrange strain value at the center of the piece:

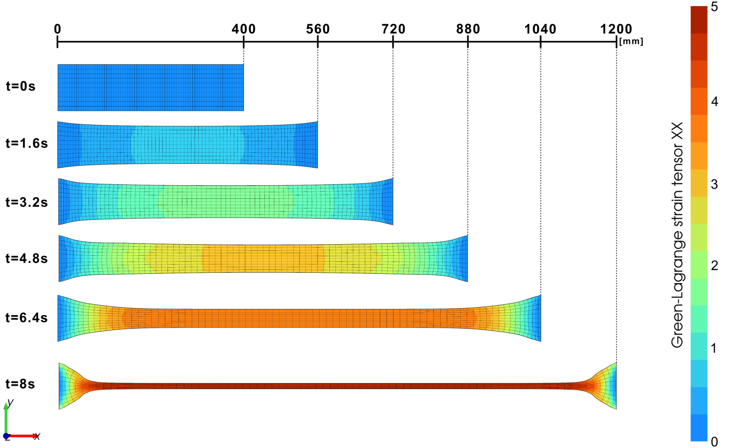




Comparison with S.Hault's results [3] in Metafor (Python user material)

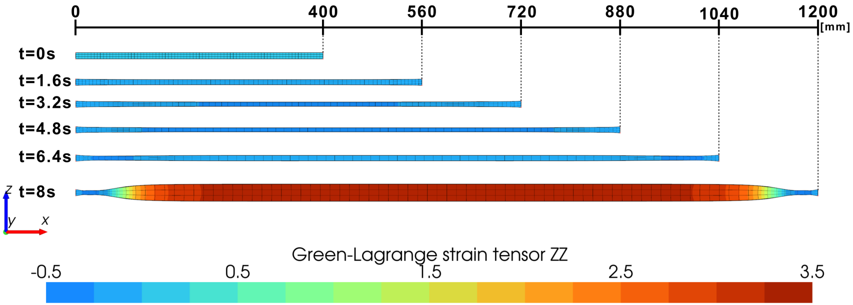
Metafor Implementation and Tests *(ctd.)*

Deformation profile:



Metafor Implementation and Tests *(ctd.)*

Deformation profile:



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